

I. V. Savelyev

FUNDAMENTALS
OF THEORETICAL
PHYSICS

Volume 1

Mechanics

Electrodynamics

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И. В. Савельев

ОСНОВЫ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

Том 1

МЕХАНИКА

ЭЛЕКТРОДИНАМИКА

ИЗДАТЕЛЬСТВО «НАУКА»

I.V. Savel'ev

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**Mechanics
Electrodynamics**

Translated from the Russian
by
G. Leib

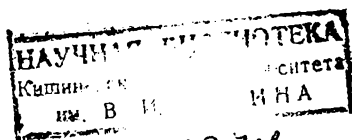
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TO THE READER

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На английском языке



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PREFACE

The book being offered to the reader is a logical continuation of the author's three-volume general course of physics. Everything possible has been done to avoid repeating what has been set out in the three-volume course. Particularly, the experiments underlying the advancing of physical ideas are not treated, and some of the results obtained are not discussed.

In the part devoted to mechanics, unlike the established traditions, Lagrange's equations are derived directly from Newton's equations instead of from d'Alembert's principle. Among the books I have acquainted myself with, such a derivation is given in A. S. Kompaneits's book *Theoretical Physics* (in Russian) for the particular case of a conservative system. In the present book, I have extended this method of exposition to systems in which not only conservative, but also non-conservative forces act.

The treatment of electrodynamics is restricted to a consideration of media with a permittivity ϵ and a permeability μ not depending on the fields \mathbf{E} and \mathbf{B} .

Sections 40 and 69 devoted to the energy-momentum tensor are appreciably more complicated. They have been included in the book because they contain an excellent illustration of how Lagrangian formalism is generalized for non-mechanical systems. A reader to whom these sections will seem too difficult may omit them without any harm to his understanding the remaining sections of the book.

I have devoted much attention to the variational principle, with the consistent use of the following procedure—first the required result is obtained with the aid of methods which the reader is already acquainted with, and then the same result is obtained using the variational principle. The object here was to ensure the reader treating the variational principle as a quite reliable and powerful means of research.

An appreciable difficulty appearing in studying theoretical physics is the circumstance that quite often many mathematical topics have either never been studied by the reader or have been forgotten by him fundamentally. To eliminate this difficulty, I have provided the book with detailed mathematical appendices. The latter are sufficiently complete to relieve the reader of having to turn to

mathematical aids and find the required information in them. This information is often set out in these aids too complicated for the readers which the present book is intended for. Hence, the information on mathematical analysis contained in a college course of higher mathematics is sufficient for mastering this book.

The book has been conceived as a training aid for students of non-theoretical specialties of higher educational institutions. I had in mind readers who would like to grasp the main ideas and methods of theoretical physics without delving into the details that are of interest only for a specialist. This book will be helpful for physics instructors at higher schools, and also for everyone interested in the subject but having no time to become acquainted with it (or restore it in his memory) according to fundamental manuals.

Igor Savelyev

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Part One

MECHANICS

Chapter I

THE VARIATIONAL PRINCIPLE IN MECHANICS

1. Introduction

Depending on the nature of the objects being studied, mechanics can be divided into particle mechanics, mechanics of a rigid body¹, and continuum mechanics. The latter, in turn, is divided into hydrodynamics, gas dynamics, the theory of elasticity, the theory of plasticity, and so on.

A continuum (continuous medium) is the most difficult object for studying in mechanics because it is a system with an infinitely great number of degrees of freedom. Besides, methods and equations of thermodynamics, electrodynamics, etc. are used in solving a number of problems treated by continuum mechanics in addition to those of theoretical mechanics. These circumstances are why continuum mechanics is the most complicated branch of mechanics. We shall not deal with topics on continuum mechanics in this book.

In the general course of physics, problems of mechanics are solved with the aid of Newton's equations. In this chapter, we shall acquaint ourselves with a different approach to a description and the studying of the motion of mechanical systems. By a mechanical system, we shall understand a collection of point particles whose motion may either be free or restricted by constraints. Particularly, a collection of point particles joined by rigid constraints forms a rigid body. In the following, for brevity's sake, we shall call point particles simply particles.

In accordance with the approach mentioned above, a function of generalized coordinates and generalized velocities of a system, and also of time, namely,

$$L = L(\text{coordinates, velocities, time})$$

called a **Lagrangian function** or a **Lagrangian** is associated with each mechanical system. Generalized coordinates q_k are defined to be any

¹ What is meant is a perfectly rigid body.

quantities by means of which the position of a system in space can be set. Generalized velocities \dot{q}_k are defined to be the time derivatives of the generalized coordinates.

The Lagrangian can be used to characterize not only systems with a finite number of degrees of freedom, but also systems with an infinite number of them—continuous media and electromagnetic and other physical fields. Thus, the significance of the Lagrangian extends beyond the scope of classical mechanics.

Having established the form of the Lagrangian for the mechanical system being considered, we can describe the motion of the system with the aid of equations relating the partial derivatives of the function L with respect to the coordinates and velocities. These equations, known as Lagrange's equations, replace Newton's equations.

The use of Lagrange's equations instead of Newton's equations has the advantage that the number of the former equals the number of degrees of freedom of the system, which, when constraints restricting the motion of the system are present, will be less than the triple number of particles in the system. The number of Newton's equations needed to describe a system of N particles, on the other hand, is $3N$. In addition, Lagrange's equations do not include the reactions of the constraints¹, which are unknown beforehand. Consequently, when using Lagrange's equations, the reactions of the constraints are automatically excluded from consideration, and this noticeably simplifies the solution of the relevant problem. True, the solution in this case gives information only on the motion of the system, the values of the reactions remaining unestablished. But in the majority of physical problems, the values of the reactions are of no interest, so that the data obtained by the method of Lagrange's equations are quite sufficient. We can indicate as an example the problem of the oscillations of a simple pendulum (Fig. 1.1). The equation of Newton's second law for the particle m has the form

$$m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{R}$$

where \mathbf{R} is the reaction of the thread. Projecting all the vectors onto axes x , y and z (the z -axis is directed beyond the drawing), we obtain three scalar equations:

$$m\ddot{x} = R_x, \quad m\ddot{y} = mg + R_y, \quad m\ddot{z} = 0$$

If we characterize the position of the system by the generalized coordinate φ , one equation will be sufficient instead of three, namely,

$$ml^2\ddot{\varphi} = -mgl \sin \varphi \quad (1.1)$$

¹ This is true only for ideal constraints, i.e. constraints without friction.

It is Lagrange's equation for the given case¹. It does not contain the reaction R . Solution of the equation yields φ as a function of t . We shall meanwhile not deal with the form of the function L for the system being considered (see Example 1 in Sec. 6).

Hence, the motion of a mechanical system can be described with the aid of either Newton's or Lagrange's equations. Naturally, the latter can be arrived at proceeding from Newton's equations (this will be done in Sec. 4). A very significant circumstance, however, is that Lagrange's equations can be obtained with the aid of a quite general variational principle—the **principle of least action**. It can be used as the foundation of classical mechanics instead of Newton's laws. A merit of the principle of least action is that it can readily be extended to systems that are not mechanical or purely mechanical, for example, elastic media, electromagnetic fields, and fields of elementary particles.

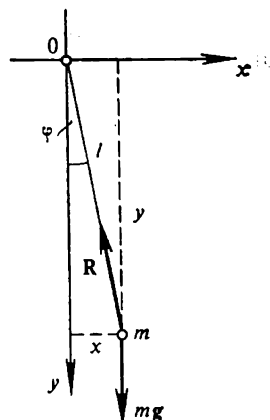


Fig. 1.1.

Summarizing, the approach to the studying of mechanical systems set out in the present chapter can be said to be much more general than the method based on Newton's laws.

2. Constraints

For a system of particles with the masses $m^{(1)}, m^{(2)}, \dots$, the equations of Newton's second law can be written as follows:

$$m_i \ddot{x}_i = F_i \quad (i = 1, 2, \dots, n) \quad (2.1)$$

where $m_1 = m_2 = m_3 = m^{(1)}$ is the mass of the 1st particle, $m_4 = m_5 = m_6 = m^{(2)}$ is that of the 2nd particle, \dots , x_1 is the coordinate x of the 1st particle, x_2 is the coordinate y of the 1st particle, x_3 is the coordinate z of the 1st particle, x_4, x_5, x_6 are the Cartesian coordinates x, y, z of the 2nd particle, \dots , F_1 is the projection onto the x -axis of the resultant force $\mathbf{F}^{(1)}$ acting on the 1st particle. F_2 is the projection of the force $\mathbf{F}^{(1)}$ onto the y -axis, F_3 is the projection of the same force onto the z -axis, F_4, F_5, F_6 are the x -, y -, z -components of

¹ The generalized velocity enters this equation in the form $\frac{d}{dt} \dot{\varphi}$. We remind our reader that dots over a symbol stand for the time derivative: $\dot{x} = dx/dt$, and $\ddot{x} = d^2x/dt^2$.

the force $F^{(2)}$ acting on the second particle, etc. The number n of equations contained in (2.1) equals the triple number of particles in the system.

Restrictions of a geometric or kinematic nature may be imposed on the positions and velocities of a system's particles. These restrictions are called constraints. *Equation, restriction of equation, condition, condition*

Examples of systems with geometric constraints are: *geometric, condition*

- (1) a particle which in its motion cannot leave a given surface or a given curve. The surface or curve may be stationary (a stationary constraint) or move in a preset way (a non-stationary constraint);
- (2) two particles A and B joined by a rigid weightless rod of length l . In this case, the restriction imposed by the constraint can be written in the form of the equation

$$(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 = l^2 \quad (2.2)$$

- (3) two particles joined by a weightless thread of length l . The analytical expression of such a constraint is

$$(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 \leq l^2 \quad (2.3)$$

- (4) a perfectly rigid body which can be considered as a system of particles with unchanging distances between them, i.e. experiencing constraints like (2.2).

An example of a system with a constraint of both a geometric and a kinematic nature is a ball rolling without slipping over a rough surface. The kinematic restriction is that the velocity of the point of contact must be zero.

In the general case, a geometric constraint can be represented by the equation

$$f(x_1, x_2, \dots, x_n, t) = 0 \quad (2.4)$$

($n = 3N$, where N is the number of particles in the system).

When restrictions are imposed on the velocities of particles in addition to their coordinates, the equation of a constraint is

$$\Phi(x_1, x_2, \dots, x_n, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n, t) = 0 \quad (2.5)$$

If Eq. (2.5) can be integrated over time, it is evidently equivalent to an equation in the form of (2.4).

Constraints of the kind given by (2.4) and integrable constraints given by (2.5) that can be reduced to them are known as **holonomic** (or **integrable**) constraints. Systems with such constraints are also called **holonomic**. The systems in Examples 1, 2, and 4 treated above belong to the holonomic type.

Hence, with holonomic constraints, the restrictions they impose are expressed in the form of equations relating the coordinates of the particles and time [see Eq. (2.4)].

Non-integrable constraints in the form of (2.5), and also constraints expressed as inequalities (see Example 3), are called **non-holonomic**.

Constraints that do not change with time are known as **stationary** (or **scleronomous**). Constraints that change with time are known as **non-stationary** (or **rheonomous**).

The equation of a holonomic stationary constraint is

$$f(x_1, x_2, \dots, x_n) = 0 \quad (2.6)$$

It differs from Eq. (2.4) for a holonomic non-stationary constraint in that it does not include the time t explicitly.

There are no general methods of solutions for problems with non-holonomic constraints. An individual approach to each problem is required. We shall not consider non-holonomic systems.

Every holonomic constraint, i.e. every constraint expressed by Eq. (2.4) or (2.6), allows us to represent one of the coordinates as a function of the others. Consequently, every such constraint diminishes the number of independent coordinates by one.

Recall that the number of independent quantities needed to determine the position of a system in space is called the **number of degrees of freedom** of the system. We can therefore say that every holonomic constraint diminishes the number of degrees of freedom of the system by one.

If constraints are absent, a system of N particles has $n = 3N$ degrees of freedom. When there are r constraints, the number of degrees of freedom will be $s = n - r = 3N - r$.

Constraints act on the particles of a system with the forces $\mathbf{R}^{(\alpha)}$ called **reactions**. A constraint with no friction is said to be **ideal**. If an ideal constraint is also stationary, its reaction \mathbf{R} is always perpendicular to the direction of the possible elementary displacement of the particle which the force \mathbf{R} is applied to (for instance, in Fig. 1.1 the reaction \mathbf{R} is directed along the thread, while the velocity of the particle is perpendicular to it). This is why reactions of ideal stationary constraints do not work on a system. If a constraint (even an ideal one) depends on the time, the work done by its reaction will be non-zero, as a rule.

The expression for the elementary work done on the particles of a system by the reactions of the constraints is $dA = \sum R_i dx_i$. We have seen that this work is zero for a system with ideal stationary constraints. Consequently, for such systems

$$\sum_i R_i dx_i = 0 \quad (2.7)$$

where dx_i are the projections onto the coordinate axes of the displacements of the particles allowed by the constraints imposed on them¹.

3. Equations of Motion in Cartesian Coordinates

Consider a system formed by interacting particles. Assume that external forces also act on the particles. We shall use the same symbols as we did in the preceding section (see the first paragraph of Sec. 2) for the coordinates of the particles and the components of the forces.

In some cases, the forces acting on particles (or at least a part of these forces) can be represented in the form

$$F_i = - \frac{\partial U}{\partial x_i} \quad (3.1)$$

where

$$U = U(x_1, x_2, \dots, x_n, t) \quad (3.2)$$

is a function of the coordinates of the particles and the time known as the **potential** of the system. If the function U does not include the time t explicitly, it is the **potential energy** of the system.

A force determined by formula (3.1) is called a **potential force**. Stationary and non-stationary potential forces should be distinguished. Forces depending only on the coordinates of a particle and not depending on the time explicitly are called **stationary**. The function U not containing the time explicitly corresponds to these forces. When U contains t explicitly, the force depends not only on the coordinates, but also on the time and, consequently, will be **non-stationary**. Stationary potential forces are called **conservative**. Systems in which only conservative forces act are also called **conservative**.

We shall note that in accordance with (3.1), the force acting on the particle numbered α can be represented as a gradient of the function (3.2):

$$F_\alpha = -\nabla_\alpha U(x_i, t) \quad (3.3)$$

Here ∇_α is an operator whose components equal the partial derivatives with respect to the coordinates of the α -th particle. By x_i here and below in the symbols of functions we denote the set of all the coordinates: x_1, x_2, \dots, x_n .

¹ In the sums encountered in theoretical physics, the index over which summation is performed (the dummy index) is repeated twice, as a rule. In this connection, it is customary practice to omit the symbol \sum and understand summation over the twice repeated indices. For instance, simply $a_i b_i$ is written instead of $\sum a_i b_i$. Although this way of writing is distinguished for its brevity, however, we shall not use it. Summation will be understood only where the symbol \sum is written.

Formula (3.3) can also be written as follows:

$$\mathbf{F}_\alpha = -\frac{\partial U}{\partial \mathbf{r}_\alpha} \quad (3.4)$$

where \mathbf{r}_α is the position vector of the particle¹.

Assume that part of the forces exerted on the particles of a system are potential and part of them are non-potential. Equation (2.1) can therefore be written as

$$m_i \ddot{\mathbf{x}}_i = -\frac{\partial U}{\partial \mathbf{x}_i} + \mathbf{F}_i^* \quad (i=1, 2, \dots, n) \quad (3.5)$$

where \mathbf{F}_i^* are the components of the non-potential forces, and n is the triple number of particles.

Equation (3.5) can be written in a form very convenient for generalizations. For this purpose, the Lagrangian mentioned in Sec. 1 is introduced. It is determined as follows for the system of particles we are considering:

$$L(\mathbf{x}_i, \dot{\mathbf{x}}_i, t) = \sum_i \frac{m_i \dot{\mathbf{x}}_i^2}{2} - U(\mathbf{x}_i, t) \quad (3.6)$$

Time differentiation of the partial derivative of L with respect to $\dot{\mathbf{x}}_i$ yields the left-hand side of Eq. (3.5):

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{x}}_i} = \frac{d}{dt} (m_i \dot{\mathbf{x}}_i) = m_i \ddot{\mathbf{x}}_i$$

The partial derivative of L with respect to \mathbf{x}_i gives the i -th component of the potential force:

$$\frac{\partial L}{\partial \mathbf{x}_i} = -\frac{\partial U}{\partial \mathbf{x}_i}$$

Consequently, we arrive at the relations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{x}}_i} - \frac{\partial L}{\partial \mathbf{x}_i} = \mathbf{F}_i^* \quad (i=1, 2, \dots, n) \quad (3.7)$$

named **Lagrange's equations**.

For systems in which only potential forces act, Lagrange's equations are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{x}}_i} - \frac{\partial L}{\partial \mathbf{x}_i} = 0 \quad (i=1, 2, \dots, n) \quad (3.8)$$

¹ By the derivative of the scalar φ with respect to the vector \mathbf{a} is understood a vector having the components $\partial\varphi/\partial a_x$, $\partial\varphi/\partial a_y$, $\partial\varphi/\partial a_z$. Consequently, the symbol $\partial\varphi/\partial \mathbf{r}$ stands for a vector with the components $\partial\varphi/\partial x$, $\partial\varphi/\partial y$, $\partial\varphi/\partial z$ (i.e. grad φ or $\nabla\varphi$; see Appendix XI).

Lagrange's equations can sometimes be written in the form of (3.8) even when the forces acting on particles depend on the velocities (the Lorentz force is an example of such forces). This can be done provided that such forces can be obtained from a certain function

$$U^*(x_i, \dot{x}_i, t) \text{ with the aid of the relation} \quad F_i = -\frac{\partial U^*}{\partial x_i} + \frac{d}{dt} \frac{\partial U^*}{\partial \dot{x}_i} \quad (3.9)$$

(we leave it to the reader to verify this statement).

The function $U^*(x_i, \dot{x}_i, t)$ is known as the **generalized potential**. We shall apply the adjective **generalized-potential** to forces corresponding to this potential. When these forces are present, the Lagrangian is written as follows:

$$L(x_i, \dot{x}_i, t) = \sum_i \frac{m_i \dot{x}_i^2}{2} - U(x_i, t) - U^*(x_i, \dot{x}_i, t) \quad (3.10)$$

The forces depending on the velocities of particles also include **dissipative forces**. This adjective is used to designate forces always directed oppositely to the velocities of particles and, consequently, causing their deceleration. Dissipative forces include, for example, forces of friction. When dissipative forces are present, the total mechanical energy of a system diminishes (dissipates), transforming into other non-mechanical kinds of energy (for instance, into the internal energy of bodies or, as we customarily say, into heat).

The dissipative forces $F_i^{(d)}$ are often proportional to the velocities of particles so that their components along the coordinate axes are determined by the equation

$$F_i^{(d)} = -k_i \dot{x}_i \quad (i=1, 2, \dots, n) \quad (3.11)$$

In this case, the dissipative forces can be expressed in terms of Rayleigh's **dissipative function**¹ equal to

$$D = \frac{1}{2} \sum_i k_i \dot{x}_i^2 \quad (3.12)$$

Indeed, comparing expressions (3.11) and (3.12), we can easily see that

$$F_i^{(d)} = -\frac{\partial D}{\partial \dot{x}_i} \quad (3.13)$$

Substituting this expression into formula (3.7) for F_i^* and assuming that there are no other non-potential forces, we obtain the equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} + \frac{\partial D}{\partial \dot{x}_i} = 0 \quad (3.14)$$

¹ This function is customarily designated by the letter F or R . To avoid confusion, however, we have preferred the symbol D .

The function D has a simple physical meaning. The work done by dissipative forces during the time dt is

$$dA = \sum_i F_i^{(d)} dx_i = \sum_i F_i^{(d)} \dot{x}_i dt = - \sum_i \frac{\partial D}{\partial \dot{x}_i} \dot{x}_i dt = -2D dt$$

(we have used Euler's theorem on homogeneous functions, see Appendix II). This work is done at the expense of the system's store of energy. Hence, the quantity $-dA/dt = 2D$ gives the rate of energy dissipation.

Lagrange's equations are superior to Newton's equations (3.5) in that, as will be shown in the following section, they retain their form when we transform from Cartesian to any generalized coordinates. When we pass over to independent generalized coordinates, the reactions of the ideal holonomic constraints vanish from the equations, which greatly facilitates the solution of problems.

4. Lagrange's Equations in Generalized Coordinates

Generalized coordinates q_k are defined to be any quantities (lengths, angles, areas, etc.) that determine the position of a mechanical system in space. As an example, we can indicate the spherical coordinates of a particle: r, ϑ, φ . Cartesian coordinates are obviously a particular case of generalized coordinates.

The time derivatives of the generalized coordinates, i.e. the quantities \dot{q}_k , are known as the **generalized velocities** of a system.

The number of independent generalized coordinates needed to set the position of a system equals the number of its degrees of freedom. In the following, we shall always choose the generalized coordinates so that their number coincides with that of the degrees of freedom of a system (i.e. so that they are all independent of one another).

We must note that generalized coordinates are often helpful in systems without constraints too (in this case their number coincides with that of the Cartesian coordinates). For instance, in solving a problem involving the motion of a particle in a central field of forces, the spherical coordinates r, ϑ, φ are more convenient than the Cartesian coordinates x, y, z .

The following representation is very helpful. Let us introduce into consideration a system of coordinates in an imaginary s -dimensional space (it is called a **configuration space** or a q -space). We plot the values of the coordinates $q_k(t)$ along the axes of this system. Hence, for each instant t , a point in the configuration space corresponds to the position of the system in conventional space. The motion of a point in our imaginary s -dimensional space corresponds to the motion of the system in the real three-dimensional space.

We shall prove that Eqs. (3.7) remain true in a transition from the Cartesian coordinates x_i to the generalized coordinates q_k , and also that Lagrange's equations written in independent generalized coordinates contain no constraint reactions.

Consider a holonomic system with ideal stationary constraints. We shall divide the non-potential forces acting on the system's particles into two categories: the constraint reactions R_i and the other non-potential forces F_i^* . Equations (3.7) will therefore become

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = R_i + F_i^* \quad (i = 1, 2, \dots, n) \quad (4.1)$$

Let r constraints expressed by the conditions

$$f_l(x_1, x_2, \dots, x_n) = 0 \quad (l = 1, 2, \dots, r) \quad (4.2)$$

be imposed on the system (since the constraints are stationary, the function f does not contain the time explicitly).

The Cartesian coordinates x_i determining the position of a system's particles can be represented as functions of the generalized coordinates q_k . If the equations of the constraints do not contain t explicitly, we can always choose q_k so that the functions expressing x_i in terms of q_k do not contain t , i.e. so that these functions are in the form

$$x_i = x_i(q_1, q_2, \dots, q_s) \quad (i = 1, 2, \dots, n) \quad (4.3)$$

(s is the number of degrees of freedom equal to $n - r$). In the following, for brevity's sake, we shall write expressions of the kind $x_i(q_1, q_2, \dots, q_s)$ in the form $x_i(q_k)$.

In accordance with (4.3), the time derivatives of the functions $x_i(q_k)$ have the form

$$\dot{x}_i = \sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k \quad (4.4)$$

Summation is performed over all the q_k 's, i.e. the subscript k takes on all the values from 1 to s . Expression (4.4) can be written for any i from 1 to n . In the following, when this will not lead to misunderstandings, we shall not indicate the values taken on by the subscript for which summation is being performed. This subscript is called a dummy index. We must note that in summation formulas a dummy index may be designated by any letter—the use of one subscript instead of another does not change the sum. Particularly, expression (4.4) can be written just as successfully, for instance, in the form

$$\dot{x}_i = \sum_l \frac{\partial x_i}{\partial q_l} \dot{q}_l \quad (4.5)$$

The quantities $\partial x_i / \partial q_h$ do not contain the generalized velocities \dot{q}_h . We can therefore state on the basis of (4.4) that

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_h} = \frac{\partial x_i}{\partial q_h} \quad (4.6)$$

We must also note that since functions (4.3) do not contain the quantities \dot{q}_h , the partial derivatives of x_i with respect to these quantities are zero:

$$\frac{\partial x_i}{\partial \dot{q}_h} = 0 \quad (4.7)$$

Finally we shall obtain another relation that we shall need later on. The quantities $\partial x_i / \partial q_h$ are functions only of the coordinates q_h (we have already noted that these quantities do not contain the velocities \dot{q}_h). Hence,

$$\frac{d}{dt} \frac{\partial x_i}{\partial q_h} = \sum_l \frac{\partial}{\partial q_l} \left(\frac{\partial x_i}{\partial q_h} \right) \dot{q}_l = \sum_l \frac{\partial^2 x_i}{\partial q_l \partial q_h} \dot{q}_l \quad (4.8)$$

Now, we shall differentiate expression (4.5) with respect to q_h . Since the derivatives $\partial \dot{q}_l / \partial q_h$ equal zero, we obtain

$$\frac{\partial \dot{x}_i}{\partial q_h} = \sum_l \frac{\partial}{\partial q_h} \left(\frac{\partial x_i}{\partial q_l} \right) \dot{q}_l = \sum_l \frac{\partial^2 x_i}{\partial q_h \partial q_l} \dot{q}_l$$

Comparing this expression with (4.8), we arrive at the relation

$$\frac{\partial \dot{x}_i}{\partial q_h} = \frac{d}{dt} \frac{\partial x_i}{\partial q_h} \quad (4.9)$$

Having obtained all the required relations, let us now turn to the proof. Multiplication of both sides of each of Eqs. (4.1) by $\partial x_i / \partial q_h$ and summation of all the equations yield

$$\sum_i \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right) \frac{\partial x_i}{\partial q_h} - \sum_i \frac{\partial L}{\partial x_i} \frac{\partial x_i}{\partial q_h} = \sum_i R_i \frac{\partial x_i}{\partial q_h} + \sum_i F_i^* \frac{\partial x_i}{\partial q_h} \quad (4.10)$$

The first of the sums on the right-hand side of this expression equals zero. To prove this, we multiply it by dq_h :

$$\left(\sum_i R_i \frac{\partial x_i}{\partial q_h} \right) dq_h = \sum_i R_i \left(\frac{\partial x_i}{\partial q_h} dq_h \right) = \sum_i R_i dx_i$$

Here dx_i are the increments of the Cartesian coordinates appearing when q_h receives the increment dq_h , the other generalized coordinates remaining unchanged. According to (2.7), however, the sum $\sum_i R_i dx_i$

for any dx_i 's allowed by their constraints is zero (we remind our reader that we assume the constraints to be stationary and ideal). Hence,

$$\left(\sum_i R_i \frac{\partial x_i}{\partial q_k} \right) dq_k = 0$$

and, since $dq_k \neq 0$, we arrive at the conclusion that

$$\sum_i R_i \frac{\partial x_i}{\partial q_k} = 0 \quad (4.11)$$

The quantity

$$Q_k^* = \sum_i F_i^* \frac{\partial x_i}{\partial q_k} \quad (4.12)$$

[see the second term on the right-hand side of Eq. (4.10)] is referred to as the **generalized force**. This name is based on the grounds that the expression

$$Q_k^* dq_k = \sum_i F_i^* \left(\frac{\partial x_i}{\partial q_k} dq_k \right) = \sum_i F_i^* dx_i$$

gives the work done by the forces F_i^* in the displacements of particles due to an increase in the generalized coordinate q_k by dq_k .

Consequently, the right-hand side of Eq. (4.10) is simply Q_k^* . Let us now consider the left-hand side of this equation. We add

the sum $\sum_i \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial q_k}$ to the minuend and the subtrahend of the left-hand side. Hence, the left-hand side of (4.10) will contain the difference of the expressions

$$\sum_i \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right) \frac{\partial x_i}{\partial q_k} + \sum_i \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial q_k} \quad (4.13)$$

and

$$\sum_i \frac{\partial L}{\partial x_i} \frac{\partial x_i}{\partial q_k} + \sum_i \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial q_k} \quad (4.14)$$

Expression (4.14) is the derivative of the Lagrangian with respect to the generalized coordinate q_k , i. e. $\partial L / \partial q_k$.

With a view to (4.9), we can write expression (4.13) in the form

$$\sum_i \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right) \frac{\partial x_i}{\partial q_k} + \sum_i \frac{\partial L}{\partial \dot{x}_i} \left(\frac{d}{dt} \frac{\partial x_i}{\partial q_k} \right) = \frac{d}{dt} \sum_i \frac{\partial L}{\partial \dot{x}_i} \frac{\partial x_i}{\partial q_k}$$

Finally, substituting $\partial \dot{x}_i / \partial \dot{q}_k$ for $\partial x_i / \partial q_k$ in accordance with (4.6), we arrive at the expression

$$\frac{d}{dt} \sum_i \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial \dot{q}_k}$$

which is exactly $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k}$. Indeed, according to the rules for the differentiation of a composite function

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_i \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial \dot{q}_k} + \sum_i \frac{\partial L}{\partial x_i} \frac{\partial \dot{x}_i}{\partial \dot{q}_k}$$

but according to (4.7), all the $\partial x_i / \partial \dot{q}_k$'s equal zero so that the first of the sums vanishes. We have thus shown that expression (4.13) equals $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k}$.

Therefore, the difference between expressions (4.13) and (4.14) is simply $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$. Introducing this value into (4.10), we arrive at the equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k^* \quad (k=1, 2, \dots, s) \quad (4.15)$$

that differ from Eqs. (4.1) in containing the generalized coordinates q_k instead of the Cartesian coordinates x_i and the generalized forces Q_k^* instead of the forces F_i^* . Equations (4.15) do not contain the reactions R_i of the constraints.

If all the forces acting on the particles of a system (except for the reactions of the constraints) are potential ones, Eqs. (4.15) become

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \quad (k=1, 2, \dots, s) \quad (4.16)$$

Equations (4.15) and (4.16) are Lagrange's equations in generalized coordinates. The function $L = L(q_1, q_2, \dots, q_s, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_s)$ in them is a function obtained from (3.6) by substituting functions (4.3) and (4.4) for the quantities x_i and \dot{x}_i .

It can be shown (see Appendix I) that Eqs. (4.15) and (4.16) also remain true for holonomic systems with ideal non-stationary constraints.

We have thus proved that Lagrange's equations have the same form in both Cartesian and generalized coordinates.

Expression (3.6) for the Lagrangian in Cartesian coordinates shows that the derivative of L with respect to \dot{x}_i equals p_i —the projection

of the momentum of the relevant particle onto one of the coordinate axes:

$$\frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i = p_i \quad (4.17)$$

and the derivative of L with respect to x_i equals F_i —the projection of the potential force acting on the particle:

$$\frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i} = F_i \quad (4.18)$$

By analogy with (4.17) and (4.18), the quantity

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \quad (4.19)$$

is called the generalized momentum, and the quantity

$$Q_k = \frac{\partial L}{\partial q_k} \quad (4.20)$$

the generalized force.

With the use of these quantities, Eqs. (4.16) can be written as

$$\frac{dp_k}{dt} = Q_k \quad (4.21)$$

similar to Newton's equations

$$\frac{dp_i}{dt} = F_i$$

We must note that definition (4.20) is more general than definition (4.12) which we have introduced for non-potential forces. If we were to extend definition (4.12) to potential forces, i.e. to forces that can be written in the form $F_i = -\partial U/\partial x_i$, we would arrive at the expression

$$Q_k = \sum_i F_i \frac{\partial x_i}{\partial q_k} = -\sum_i \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial q_k} = -\frac{\partial U}{\partial q_k} \quad (4.22)$$

According to definition (4.20), however,

$$Q_k = \frac{\partial L}{\partial q_k} = \frac{\partial (T-U)}{\partial q_k} = \frac{\partial T}{\partial q_k} - \frac{\partial U}{\partial q_k} \quad (4.23)$$

Expression (4.23) differs from (4.22) in the term $\partial T/\partial q_k$ which, as will be shown in the following section, generally speaking, is non-zero [see formula (5.9)].

5. The Lagrangian and Energy

The Lagrangian $L(q_k, \dot{q}_k, t)$ is a characteristic function of a mechanical system. It is quite natural that not only momenta and forces

[see formulas (4.19) and (4.20)], but also the energy of a system can be expressed in terms of this function.

The expression for the total energy of a system in terms of the Lagrangian is

$$E = \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L(q_k, \dot{q}_k, t) \quad (5.1)$$

The grounds for determining E in exactly this way will be revealed below. Expression (5.1) is more general than the one known from the general course of physics, i.e.

$$E = T + U \quad (5.2)$$

(Here T is the kinetic energy, and U , the potential energy.) Definition (5.1) also holds when the total energy cannot be represented as the sum of the kinetic and potential energies.

Let us calculate the total time derivative of the quantity (5.1)

$$\begin{aligned} \frac{dE}{dt} &= \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k + \sum_k \dot{q}_k \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \sum_k \frac{\partial L}{\partial q_k} \dot{q}_k \\ &\quad - \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k - \frac{\partial L}{\partial t} = \sum_k \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} \right) \dot{q}_k - \frac{\partial L}{\partial t} \end{aligned}$$

In accordance with Lagrange's equation (4.15), the expression in parentheses equals the non-potential generalized force Q_k^* . Hence,

$$\frac{dE}{dt} = \sum_k Q_k^* \dot{q}_k - \frac{\partial L}{\partial t} = W^* - \frac{\partial L}{\partial t} \quad (5.3)$$

where W^* is the power of all the non-potential forces acting on the system's particles.

When the Lagrangian does not contain the time explicitly, $\partial L / \partial t = 0$, and formula (5.3) becomes

$$\frac{dE}{dt} = W^*$$

which is known from the general course of physics.

A glance at (5.3) shows that for conservation of the total energy of a system, the absence of non-potential forces is not sufficient. It is also essential that the time be absent in the Lagrangian explicitly, i.e. that the system be conservative. We have thus established that for a closed system in which only conservative forces act, the quantity determined by formula (5.1) remains constant. This is why quantity (5.1) is known as the **total energy of a system**.

The functions of the quantities q_k and \dot{q}_k that in the motion of a system retain a constant value determined by the initial conditions are called **integrals of motion** in mechanics. Consequently, the energy

of a conservative closed system can be said to be an integral of motion.

Let us find the conditions in which the definition (5.1) transforms into (5.2). To do this, we shall investigate the expression for T in generalized coordinates. We assume that the constraints are holonomic. We express the Cartesian coordinates x_i of the system's particles in terms of the generalized coordinates q_k :

$$x_i = x_i(q_1, q_2, \dots, q_s, t) \quad (i = 1, 2, \dots, n) \quad (5.4)$$

We must note that the time is contained explicitly in these relations only for non-stationary constraints.

We find the total time derivatives of the functions x_i . Since the quantities q_k are functions of t , the required derivatives have the form

$$\dot{x}_i = \frac{dx_i}{dt} = \frac{\partial x_i}{\partial t} + \sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k \quad (i = 1, 2, \dots, n)$$

We introduce these expressions into the formula for the kinetic energy:

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i \dot{x}_i^2 = \frac{1}{2} \sum_i m_i \left(\frac{\partial x_i}{\partial t} + \sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k \right)^2 \\ &= \sum_i \frac{m_i}{2} \left(\frac{\partial x_i}{\partial t} \right)^2 + \sum_i m_i \frac{\partial x_i}{\partial t} \sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k + \sum_i \frac{m_i}{2} \left(\sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k \right)^2 \end{aligned} \quad (5.5)$$

In accordance with (5.4), the expressions $\partial x_i / \partial t$ and $\partial x_i / \partial q_k$ are functions of q_k and t ; they do not contain the quantities \dot{q}_k . Consequently, the first of the sums in formula (5.5) is also a function of q_k and t . Designating this function by the letter α , we have

$$\alpha(q_k, t) = \sum_i \frac{m_i}{2} \left(\frac{\partial x_i}{\partial t} \right)^2 \quad (5.6)$$

Changing the sequence of summation over i and k in the second of the sums in formula (5.5), we give it the form

$$\sum_k \dot{q}_k \left\{ \sum_i m_i \frac{\partial x_i}{\partial t} \frac{\partial x_i}{\partial q_k} \right\} = \sum_k \beta_k(q_j, t) \dot{q}_k$$

where

$$\beta_k(q_j, t) = \sum_i m_i \frac{\partial x_i}{\partial t} \frac{\partial x_i}{\partial q_k} \quad (5.7)$$

Finally, representing one of the factors in the third term of (5.5) in the form $\sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k$, and the other factor in the form $\sum_l \frac{\partial x_i}{\partial q_l} \dot{q}_l$,

and then, changing the sequence of summation over i , k , and l , we write the third term as follows:

$$\begin{aligned} \sum_i \frac{m_i}{2} \sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k \sum_l \frac{\partial x_i}{\partial q_l} \dot{q}_l &= \sum_{k,l} \dot{q}_k \dot{q}_l \sum_i \frac{m_i}{2} \frac{\partial x_i}{\partial q_k} \frac{\partial x_i}{\partial q_l} \\ &= \sum_{k,l} \gamma_{kl}(q_j, t) \dot{q}_k \dot{q}_l \end{aligned}$$

where

$$\gamma_{kl}(q_j, t) = \sum_i \frac{m_i}{2} \frac{\partial x_i}{\partial q_k} \frac{\partial x_i}{\partial q_l} \quad (5.8)$$

The expression for the kinetic energy can thus be written as

$$T = \alpha(q_j, t) + \sum_k \beta_k(q_j, t) \dot{q}_k + \sum_{k,l} \gamma_{kl}(q_j, t) \dot{q}_k \dot{q}_l \quad (5.9)$$

With stationary constraints, t is not contained explicitly in function (5.4). Consequently, $\partial x_i / \partial t = 0$, so that the coefficients $\alpha(q_j, t)$ and $\beta_k(q_j, t)$ vanish. In this case, the coefficients γ_{kl} do not contain the time explicitly. The result is

$$T = \sum_{k,l} \gamma_{kl}(q_j) \dot{q}_k \dot{q}_l \quad (5.10)$$

Therefore, the kinetic energy of a system with stationary constraints is a homogeneous quadratic function of the generalized velocities \dot{q}_k .

We established in Sec. 3 that for a system of particles, the Lagrangian in Cartesian coordinates is

$$L(x_i, \dot{x}_i, t) = \sum_i \frac{m_i \dot{x}_i^2}{2} - U(x_i, t)$$

In the transition from Cartesian to generalized coordinates, the first term on the right-hand side of the above equation transforms, generally speaking, into expression (5.9). If we limit ourselves to a consideration of stationary constraints, however, the first term must be replaced with expression (5.10). The second term becomes $U(q_k, t)$. With stationary constraints, the time is contained explicitly in U provided that the potential forces are non-stationary.

We have thus established that for stationary constraints the Lagrangian of a particle system written in generalized coordinates is

$$L(q_k, \dot{q}_k, t) = T(q_k, \dot{q}_k) - U(q_k, t)$$

where T is determined by formula (5.10). Using this expression in formula (5.1), we get

$$E = \sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k - T(q_k, \dot{q}_k) + U(q_k, t) \quad (5.11)$$

In accordance with (5.10), T is a homogeneous quadratic function of the quantities \dot{q}_k . Consequently, the following relation holds for it on the basis of Euler's theorem (see Appendix II):

$$\sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$$

Making such a substitution in (5.11), we arrive at the formula

$$E = 2T - T + U = T(q_k, \dot{q}_k) + U(q_k, t)$$

We obtained this formula with only a single assumption, namely, that the constraints are stationary. If, in addition, the potential forces are also stationary, the potential $U(q_k, t)$ transforms into the potential energy $U(q_k)$, and formula (5.1) transforms into (5.2).

Definition (5.1) thus coincides with (5.2) when both the constraints and the potential forces are stationary.

6. Examples of Compiling Lagrange's Equations

Having selected the independent generalized coordinates convenient for describing the system being considered, we must establish the form of the functions

$$x_i = x_i(q_k, t) \text{ and } \dot{x}_i = \dot{x}_i(q_k, \dot{q}_k, t)$$

and introduce them instead of the quantities x_i and \dot{x}_i into the expression for the Lagrangian

$$L = T(\dot{x}_i) - U(x_i, t) = \sum_i \frac{m_i}{2} \dot{x}_i^2 - U(x_i, t)$$

The result is the Lagrangian in generalized coordinates:

$$L = L(q_k, \dot{q}_k, t) = T(q_k, \dot{q}_k, t) - U(q_k, t)$$

If the expression obtained for L contains terms that do not depend on q_k and \dot{q}_k , they may be discarded because they do not contribute to the quantities $\partial L / \partial q_k$ and $\partial L / \partial \dot{q}_k$ and, consequently, do not affect the form of Lagrange's equations.

We sometimes succeed in greatly simplifying the operation of finding the function $T(q_k, \dot{q}_k, t)$. This is possible when it is easy to

establish the relation between a particle's elementary displacement ds and the increments of the generalized coordinates q_k . For instance, in polar coordinates on a plane (Fig. 6.1), the displacement ds is a diagonal of a rectangle constructed on the sides dr and $r d\varphi$ (we must

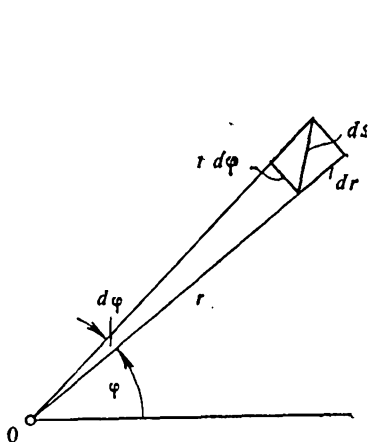


Fig. 6.1.

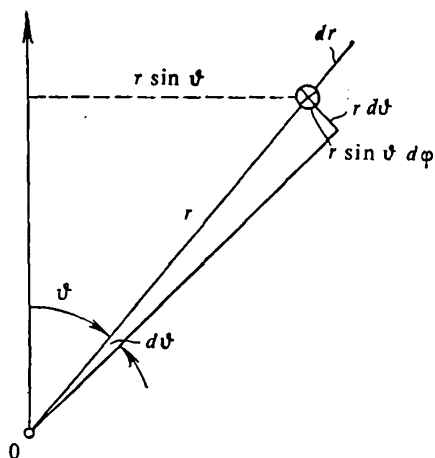


Fig. 6.2.

remember the smallness of $d\varphi$). Consequently, $ds^2 = dr^2 + r^2 d\varphi^2$. Dividing this quantity by dt^2 , we obtain the square of the velocity of a particle: $v^2 = \dot{r}^2 + r^2 \dot{\varphi}^2$. Finally,

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) \quad (6.1)$$

For cylindrical coordinates, a third coordinate z is added to the previous two (r and φ). The displacement ds is a diagonal of a rectangular parallelepiped with the sides dr , $r d\varphi$, and dz . Hence,

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{z}^2) \quad (6.2)$$

Three mutually perpendicular segments (Fig. 6.2) of lengths dr , $r d\varphi$, $r \sin \vartheta d\varphi$ (the last segment is directed beyond the drawing¹) correspond to the increments of the polar coordinates r , ϑ , φ . The displacement ds coincides with a diagonal of a rectangular parallelepiped constructed on these segments. Therefore,

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\vartheta}^2 + r^2 \sin^2 \vartheta \dot{\varphi}^2) \quad (6.3)$$

¹ We shall depict segments perpendicular to the plane of the drawing by a circle with a dot if the segment is directed towards us, and by a circle with a cross if the segment is directed behind the drawing.

Let us consider several examples.

1. A Simple Pendulum (see Fig. 1.1). The displacement of the particle m is $ds = l d\varphi$. Hence, $T = \frac{1}{2} ml^2 \dot{\varphi}^2$. The potential energy is $U = -mgl \cos \varphi$. Consequently, the Lagrangian is

$$L = \frac{1}{2} ml^2 \dot{\varphi}^2 + mgl \cos \varphi$$

We invite our reader to write Lagrange's equation and convince himself that it coincides with Eq. (1.1).

We find the generalized momentum and generalized force. By formula (4.19)

$$p_{\varphi} = \frac{\partial L}{\partial \dot{\varphi}} = ml^2 \dot{\varphi} = mvl$$

Hence, in the given case, the generalized momentum coincides with the moment of the conventional momentum mv (the angular momentum) relative to the point of suspension of the pendulum.

By formula (4.20),

$$Q_{\varphi} = \frac{\partial L}{\partial \varphi} = -mgl \sin \varphi$$

and it is exactly the moment N of the force mg relative to the point of suspension. We must note that the elementary work is

$$dA = Q_{\varphi} d\varphi = N d\varphi$$

The last expression coincides with that for the work in rotation known from elementary mechanics.

Finally, we find the energy of the pendulum by formula (5.1) (we can do this in the given case because the constraint is stationary):

$$E = \frac{\partial L}{\partial \dot{\varphi}} \dot{\varphi} - L = ml^2 \dot{\varphi}^2 - \frac{1}{2} ml^2 \dot{\varphi}^2 - mgl \cos \varphi = \frac{1}{2} m (l \dot{\varphi})^2 - mgl \cos \varphi$$

2. A Pendulum with a Uniformly Moving Point of Suspension. Assume that the point of suspension of a simple pendulum moves in a horizontal direction at the constant velocity v in the plane of oscillations of the pendulum (Fig. 6.3). The equation of the constraint is

$$f(x, y, t) = (x - vt)^2 + y^2 - l^2 = 0$$

(the constraint is non-stationary).

From the expressions for the Cartesian coordinates

$$x = l \sin \varphi + vt, \quad y = l \cos \varphi$$

it follows that

$$\dot{x} = (l \cos \varphi) \dot{\varphi} + v, \quad \dot{y} = -(l \sin \varphi) \dot{\varphi}$$

whence

$$T = \frac{1}{2} m [l^2 \dot{\varphi}^2 + (2vl \cos \varphi) \dot{\varphi} + v^2]$$

The potential energy $U = -mgy = -mgl \cos \varphi$.

In compiling the Lagrangian, the constant term $\frac{1}{2}mv^2$ may be discarded. Consequently

$$L = \frac{1}{2} m [l^2 \dot{\varphi}^2 + (2vl \cos \varphi) \dot{\varphi}] + mgl \cos \varphi$$

Notwithstanding the non-stationary nature of the constraint, the time does not enter the function L explicitly.

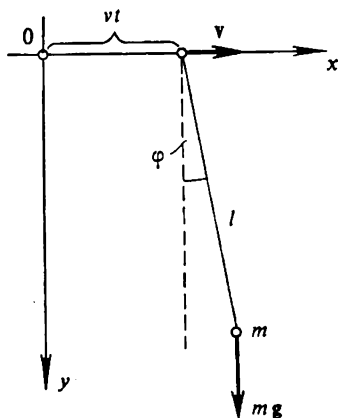


Fig. 6.3.

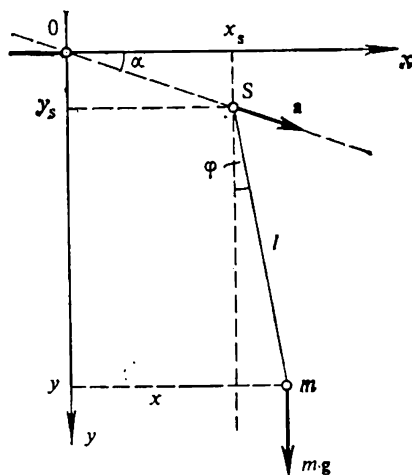


Fig. 6.4.

We invite our reader to find Lagrange's equations in this and the following examples.

3. A Pendulum with a Point of Suspension Moving with Constant Acceleration. Consider a simple pendulum whose point of suspension S moves with the constant acceleration a along a straight line making the angle α with a horizontal plane (Fig. 6.4). The coordinates of the point of suspension are

$$x_s = \left(\frac{1}{2} a \cos \alpha \right) t^2, \quad y_s = \left(\frac{1}{2} a \sin \alpha \right) t^2$$

and the coordinates of point m are

$$x = \left(\frac{1}{2} a \cos \alpha \right) t^2 + l \sin \varphi, \quad y = \left(\frac{1}{2} a \sin \alpha \right) t^2 + l \cos \varphi$$

Differentiation of x and y with respect to t yields

$$\dot{x} = (a \cos \alpha) \dot{t} + (l \cos \varphi) \dot{\varphi}, \quad \dot{y} = (a \sin \alpha) \dot{t} - (l \sin \varphi) \dot{\varphi}$$

Introducing these values of \dot{x} and \dot{y} into the expression for the kinetic energy, we obtain

$$T = \frac{1}{2} m a^2 \dot{t}^2 + m a l (\cos \alpha \cos \varphi - \sin \alpha \sin \varphi) \dot{\varphi} \dot{t} + \frac{1}{2} m l^2 \dot{\varphi}^2$$

The potential energy is $U = -mgy = -mg \left[\left(\frac{1}{2} a \sin \alpha \right) t^2 + l \cos \varphi \right]$. In writing the expression for the Lagrangian, we may omit the terms $\frac{1}{2} m a^2 \dot{t}^2$ in T and $-\left(\frac{1}{2} m g a \sin \alpha \right) t^2$ in U because they do not contain φ and $\dot{\varphi}$ and therefore cannot affect the form of Lagrange's equations. As a result, we find that

$$L = m a l (\cos \alpha \cos \varphi - \sin \alpha \sin \varphi) \dot{\varphi} \dot{t} + \frac{1}{2} m l^2 \dot{\varphi}^2 + m g l \cos \varphi$$

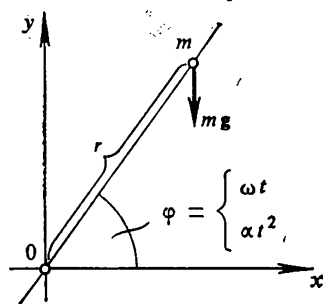


Fig. 6.5.

The Lagrangian depends explicitly on t (this is due to the non-stationary nature of the constraint), the time having entered L through the agency of the kinetic energy T .

4. A Particle Moving along a Uniformly Rotating Straight Line. Assume that a particle of mass m ex-

periences a non-stationary constraint consisting in that the particle can move only along a straight line rotating at a constant angular velocity ($\varphi = \omega t$) in a vertical plane (Fig. 6.5). In addition to the reaction of the constraint, the particle is acted upon by the potential force mg . The Cartesian coordinates of the particle are

$$x = r \cos \omega t, \quad y = r \sin \omega t$$

Consequently,

$$\dot{x} = \dot{r} \cos \omega t - r \omega \sin \omega t, \quad \dot{y} = \dot{r} \sin \omega t + r \omega \cos \omega t$$

The kinetic energy is

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \omega^2)$$

The potential energy is

$$U = mgy = mgr \sin \omega t$$

(when we passed from the Cartesian coordinates to the generalized coordinate r , the time t entered the expression for U explicitly).

Finally, the Lagrangian is

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \omega^2) - mgr \sin \omega t$$

It depends explicitly on t ; now the time has entered L in terms of the potential energy U .

The generalized momentum is

$$p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} = mv$$

where v is the particle's velocity along the straight line.

The generalized force

$$Q_r = \frac{\partial L}{\partial r} = mr\omega^2 - mg \sin \omega t$$

consists of two terms. The first of them, $mr\omega^2$, is the centrifugal force of inertia, and the second, $-mg \sin \omega t$, is the projection of the force mg onto the direction r .

5. **A Particle Moving along a Straight Line Rotating with Acceleration.** Assume that the straight line giving rise to a constraint (see Fig. 6.5) rotates not uniformly, but with acceleration ($\varphi = \alpha t^2$). Hence,

$$x = r \cos \alpha t^2, \quad y = r \sin \alpha t^2$$

$$\dot{x} = \dot{r} \cos \alpha t^2 - r 2\alpha t \sin \alpha t^2$$

$$\dot{y} = \dot{r} \sin \alpha t^2 + r 2\alpha t \cos \alpha t^2$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 4\alpha^2 t^2)$$

$$U = mgy = mgr \sin \alpha t^2$$

Accordingly,

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 4\alpha^2 t^2) - mgr \sin \alpha t^2$$

In this example, the time entered L explicitly in terms of both T and U .

7. Principle of Least Action¹

Instead of Newton's laws, mechanics can be based on the principle of least action or the Hamilton principle². The action S during the

¹ Before beginning to study this section, the reader should acquaint himself with Appendix III.

² This principle was established by the Irish mathematician William Hamilton in 1834.

time interval from t_1 to t_2 is defined to be the integral

$$S = \int_{t_1}^{t_2} L(q_k, \dot{q}_k, t) dt \quad (7.1)$$

where $L(q_k, \dot{q}_k, t)$ is the Lagrangian of the system being considered. Integration is performed from the instant t_1 at which the position of the system is characterized by the values of the coordinates $q_k(t_1)$ to the instant t_2 at which the position of the system is determined by the values of the coordinates $q_k(t_2)$.

According to the principle of least action, a system moves between the positions $q_k(t_1)$ and $q_k(t_2)$ in such a way [i.e. the functions $q_k(t)$ have such a form] that the action (7.1) has the smallest possible value. Using the notion of configuration space, we can say that a point depicting the position of a system moves in this space along the curve for which the action S is minimum.

The Hamilton principle is the most general formulation of the law of motion of mechanical systems. A merit of this formulation is that it can easily be extended to systems that are not purely mechanical, for instance to elastic media and electromagnetic fields.

Inspection of (7.1) shows that the dimension of action equals that of the product of energy and time (or momentum and displacement). Planck's constant, also called a quantum of action, has the same dimension.

It is a simple matter to obtain Lagrange's equations from the principle of least action. The action S is a functional (see Appendix III). According to the calculus of variations, a functional reaches an extreme value provided that its variation is zero. Consequently, the principle of least action can be expressed by the condition

$$\delta S = \delta \int_{t_1}^{t_2} L(q_k, \dot{q}_k, t) dt = 0 \quad (7.2)$$

For this reason, the principle of least action is known as the **variational principle of mechanics**.

It is known from the calculus of variations (see Appendix III) that the variation of a functional of the type (7.1)¹ vanishes if we take functions satisfying Euler's equations (III.25) as $q_k(t)$. In the given case, Euler's equations have the form

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = 0 \quad (k = 1, 2, \dots, s) \quad (7.3)$$

i.e. coincide with Lagrange's equations.

¹ Compare with (III.20). In the given case, the role of $f(x, y_k, \dot{y}_k)$ is played by $L(t, q_k, \dot{q}_k)$. The role of the independent variable x is played by t , that of $y_k(x)$ is played by $q_k(t)$, and that of $\dot{y}_k(x)$, by the function $\dot{q}_k(t)$.

We have thus convinced ourselves that the Hamilton principle leads to Lagrange's equations.

Appendix III shows [see the text following formula (III.19)] that the addition to the integrand in (7.2) of the total derivative with respect to t of any function of the generalized coordinates and time does not change the conditions of the extremum, i.e. Eqs. (7.3). Consequently, the Lagrangian must be determined to within the additive summands that are the total time derivative of an arbitrary function of the generalized coordinates and time (a constant is a particular case of such a function). We have already noted this circumstance in Sec. 4 and used it in Sec. 6.

CONSERVATION LAWS

8: Energy Conservation

The conservation laws considered in mechanics are based on the properties of space and time. Energy conservation is associated with the homogeneity of time, momentum conservation—with the homogeneity of space, and, finally, angular momentum conservation is associated with the isotropy of space¹.

We shall begin with the law of energy conservation. Assume that a system of particles is in unchanging external conditions (this occurs if the system is closed or experiences the action of a constant external force field); the constraints (if present) are ideal and stationary. In this case, the time, owing to its homogeneity, cannot enter the Lagrangian explicitly. Indeed, homogeneity signifies the equivalence of all instants. Therefore, the replacement of one instant with another without changing the values of the coordinates and velocities of the particles should not alter the mechanical properties of the system. This is naturally true only if the replacement of one instant with another does not change the conditions in which the system is, i.e. if the external field is time independent (particularly, this field may be absent).

Hence, for a closed system or one in a stationary force field, we have $\partial L / \partial t = 0$. Therefore,

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k \quad (8.1)$$

If a system is conservative, the motion of the particles obeys Eqs. (4.16). In accordance with these equations, we shall substitute $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k}$ for $\frac{\partial L}{\partial q_k}$. Expression (8.1) can therefore be written as

¹ Homogeneity signifies identical properties at all points. Isotropy signifies identical properties at each point in all directions. Homogeneity and isotropy are independent of each other. A medium may have different properties at different points, the properties at one point differing from those at others, but being the same in all directions. Such a medium will be non-homogeneous, but isotropic. A medium is possible whose properties are the same at all points, but differ (in the same way at all points) in different directions. Such a medium will be homogeneous, but anisotropic. It is quite obvious that there may be homogeneous and isotropic media, and also non-homogeneous and anisotropic ones

follows:

$$\frac{dL}{dt} = \sum_k \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} \dot{q}_k = \frac{d}{dt} \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k$$

The latter expression can be given the form

$$\frac{d}{dt} \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - \frac{dL}{dt} = \frac{d}{dt} \left(\sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L \right) = 0$$

According to formula (5.1), the quantity in parentheses is the energy E of the system. We have thus arrived at the statement that $dE/dt = 0$, whence

$$E = \text{const} \quad (8.2)$$

Hence, the homogeneity of time leads to the following law: *the energy of a closed conservative system of particles (or a system in a stationary external force field) remains constant.*

Inspection of (5.1) shows that E is a function of the generalized coordinates q_k and the generalized velocities \dot{q}_k . Functions of the quantities q_k and \dot{q}_k that retain in motion a constant value determined by the initial conditions are called **integrals of motion**. Consequently, the energy of a closed system is an integral of motion.

9. Momentum Conservation

Consider a closed system of particles. Closure signifies that the action of external bodies on the system's particles is negligibly small. Owing to the homogeneity of space, the displacement of all the particles of the system through an identical length $\delta \mathbf{r}$ must not change the mechanical properties of the system—the Lagrangian must retain its previous value. For an unclosed system, such a translation would cause a change in the arrangement of the particles relative to the bodies interacting with them, which would affect the system's mechanical properties. Therefore, we may state only for a closed system of particles that the parallel translation of the system as a whole is not attended by a change in the function L (i.e. $\delta L = 0$).

Assuming the displacement $\delta \mathbf{r}$ to be very small, we can write

$$\delta L = \sum_{\alpha} \frac{\partial L}{\partial \mathbf{r}_{\alpha}} \delta \mathbf{r}_{\alpha} = \delta \mathbf{r} \sum_{\alpha} \frac{\partial L}{\partial \mathbf{r}_{\alpha}} = 0^1 \quad (9.1)$$

¹ See the footnote on p. 17. In accordance with what is said in this footnote,

$$\frac{\partial \Phi}{\partial \mathbf{r}} d\mathbf{r} = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz.$$

(α is the number of the particle). We took advantage of the circumstance that the displacements $\delta \mathbf{r}_\alpha$ of the particles are the same and equal $\delta \mathbf{r}$.

According to our assumption, $\delta \mathbf{r} \neq 0$. It thus follows from (9.1) that

$$\sum_{\alpha} \frac{\partial L}{\partial \mathbf{r}_\alpha} = 0 \quad (9.2)$$

Lagrange's equations (3.8) allow us to write

$$\begin{aligned} \frac{\partial L}{\partial x_\alpha} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_\alpha} = \frac{d}{dt} \frac{\partial L}{\partial v_{\alpha x}} \\ \frac{\partial L}{\partial y_\alpha} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{y}_\alpha} = \frac{d}{dt} \frac{\partial L}{\partial v_{\alpha y}} \\ \frac{\partial L}{\partial z_\alpha} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{z}_\alpha} = \frac{d}{dt} \frac{\partial L}{\partial v_{\alpha z}} \end{aligned}$$

Multiplying the first, second, and third of these equations by the unit vectors \mathbf{e}_x , \mathbf{e}_y , and \mathbf{e}_z , respectively¹, and summing them, we obtain the expression

$$\frac{\partial L}{\partial \mathbf{r}_\alpha} = \frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}_\alpha} \quad (9.3)$$

Equation (9.2) can thus be written as

$$\frac{d}{dt} \sum_{\alpha} \frac{\partial L}{\partial \mathbf{v}_\alpha} = 0 \quad (9.4)$$

The quantity $\frac{\partial L}{\partial \mathbf{v}_\alpha}$ is a vector with the components $\frac{\partial L}{\partial v_{\alpha x}} = \frac{\partial L}{\partial \dot{x}_\alpha}$, $\frac{\partial L}{\partial v_{\alpha y}} = \frac{\partial L}{\partial \dot{y}_\alpha}$, and $\frac{\partial L}{\partial v_{\alpha z}} = \frac{\partial L}{\partial \dot{z}_\alpha}$. According to (4.17), these products are the projections of the conventional (not generalized) momentum \mathbf{p}_α of the α -th particle onto the coordinate axes. Hence,

$$\frac{\partial L}{\partial \mathbf{v}_\alpha} = \mathbf{p}_\alpha \quad (9.5)$$

With account taken of this circumstance, Eq. (9.4) will be written as follows:

$$\frac{d}{dt} \sum_{\alpha} \mathbf{p}_\alpha = 0$$

¹ Instead of the symbols \mathbf{i} , \mathbf{j} , \mathbf{k} , we shall designate the unit vectors of the coordinate axes by the symbol \mathbf{e} with the relevant subscript.

Hence it follows that

$$\mathbf{p} = \sum_{\alpha} \mathbf{p}_{\alpha} = \text{const} \quad (9.6)$$

where \mathbf{p} is the total momentum of the system.

Proceeding from the homogeneity of space, we have thus arrived at the following law: *the total momentum of a closed system of particles remains constant*. Consequently, the momentum of a closed system is also an integral of motion.

10. Angular Momentum Conservation

Owing to the isotropy of space, the mechanical properties of a closed system of particles should not change upon the arbitrary rotation of the system as a whole in space. Accordingly, the Lagrangian should also remain unchanged ($\delta L = 0$). Let us find the increment of the Lagrangian δL in an arbitrary very small rotation of a system through the angle $\delta\varphi$. All the vectors characterizing the system will rotate together with it. As a result, they will receive certain increments that will be of the same order as $\delta\varphi$. According to formula (VI.46)

$$\delta \mathbf{r}_{\alpha} = [\delta\varphi, \mathbf{r}_{\alpha}], \quad \delta \mathbf{v}_{\alpha} = [\delta\varphi, \mathbf{v}_{\alpha}] \quad (10.1)$$

Owing to the smallness of the quantities $\delta \mathbf{r}_{\alpha}$ and $\delta \mathbf{v}_{\alpha}$

$$\delta L = \sum_{\alpha} \frac{\partial L}{\partial \mathbf{r}_{\alpha}} \delta \mathbf{r}_{\alpha} + \sum_{\alpha} \frac{\partial L}{\partial \mathbf{v}_{\alpha}} \delta \mathbf{v}_{\alpha}$$

[we remind our reader that $L = L(\mathbf{r}_{\alpha}, \dot{\mathbf{r}}_{\alpha}) = L(\mathbf{r}_{\alpha}, \mathbf{v}_{\alpha})$, and L does not contain the time explicitly]. With a view to (10.1),

$$\delta L = \sum_{\alpha} \frac{\partial L}{\partial \mathbf{r}_{\alpha}} [\delta\varphi, \mathbf{r}_{\alpha}] + \sum_{\alpha} \frac{\partial L}{\partial \mathbf{v}_{\alpha}} [\delta\varphi, \mathbf{v}_{\alpha}] \quad (10.2)$$

It is known from vector algebra that a cyclic transposition of the multipliers may be performed in a scalar triple product [see formula (VI.3)]. Such a transposition in (10.2) yields

$$\delta L = \sum_{\alpha} \delta\varphi \left[\mathbf{r}_{\alpha}, \frac{\partial L}{\partial \mathbf{r}_{\alpha}} \right] + \sum_{\alpha} \delta\varphi \left[\mathbf{v}_{\alpha}, \frac{\partial L}{\partial \mathbf{v}_{\alpha}} \right]$$

Let us put $\delta\varphi$ outside the sum sign, simultaneously substituting $\frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}_{\alpha}}$ for $\frac{\partial L}{\partial \mathbf{r}_{\alpha}}$ in accordance with Lagrange's equations (9.3):

$$\delta L = \delta\varphi \left\{ \sum_{\alpha} \left[\mathbf{r}_{\alpha}, \frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}_{\alpha}} \right] + \sum_{\alpha} \left[\mathbf{v}_{\alpha}, \frac{\partial L}{\partial \mathbf{v}_{\alpha}} \right] \right\} = \delta\varphi \frac{d}{dt} \sum_{\alpha} \left[\mathbf{r}_{\alpha}, \frac{\partial L}{\partial \mathbf{v}_{\alpha}} \right]$$

According to our assumption, $\delta\varphi \neq 0$, therefore the condition $\delta L = 0$ is equivalent to the condition

$$\frac{d}{dt} \sum_{\alpha} \left[\mathbf{r}_{\alpha}, \frac{\partial L}{\partial \mathbf{v}_{\alpha}} \right] = 0$$

or

$$\sum_{\alpha} \left[\mathbf{r}_{\alpha}, \frac{\partial L}{\partial \mathbf{v}_{\alpha}} \right] = \text{const}$$

According to (9.5), $\partial L / \partial \mathbf{v}_{\alpha}$ is the conventional momentum \mathbf{p}_{α} . The quantity $\mathbf{M} = [\mathbf{r}, \mathbf{p}]$ is the angular momentum (moment of momentum) of the relevant particle with respect to the origin of coordinates. We have thus arrived at the statement that

$$\mathbf{M} = \sum_{\alpha} \mathbf{M}_{\alpha} = \sum_{\alpha} [\mathbf{r}_{\alpha}, \mathbf{p}_{\alpha}] = \text{const} \quad (10.3)$$

In this relation, \mathbf{M}_{α} is the angular momentum of the α -th particle, and \mathbf{M} is the resultant angular momentum of the system.

With a view to the isotropy of space, we have thus arrived at the following law: *the resultant angular momentum of a closed system of particles remains constant*. Consequently, the angular momentum of a closed system, like its energy and momentum, is an integral of motion.

Assume that a system of particles is in an external central force field, i.e. a field in which the force action on any of the particles has a direction passing through the same fixed point O (the centre of the field), while the magnitude of the force depends only on the distance r to the particle from this point. The potential energy of a particle in such a field is

$$U = U(r) \quad (10.4)$$

Arbitrary rotation of the system in space about the point O does not change the mechanical properties of the system (the arrangement of the particles relative to the force centre O remains constant in such rotation). Hence, although in the given case the system of particles is not closed, its angular momentum will be constant. True, this holds only for the angular momentum with respect to the point O . For a closed system, however, the angular momentum with respect to any point is conserved.

If an external field has axial symmetry (this signifies that the potential energy of a particle depends only on the distance R to the particle from the given axis), the mechanical properties of the system will not change upon rotation about the axis of the field. Therefore, the angular momentum (moment of momentum) of the system relative to this axis will be constant (we remind our reader that the moment relative to an axis is defined as the projection onto this axis of a moment taken relative to any of the points on the axis).

Chapter III

SELECTED PROBLEMS IN MECHANICS

11. Motion of a Particle in a Central Force Field

Consider the motion of a particle in a central field of the kind

$$U(r) = \frac{a}{r} \quad (11.1)$$

where a is a constant that may be either positive or negative. A positive constant corresponds to repulsion of the particle from the force centre (for instance, the Coulomb force of repulsion), and a negative one to attraction of the particle to the centre (the Coulomb force of attraction or the force of gravitational interaction of the particle with a stationary particle at the centre of the field).

We established in Sec. 10 that the angular momentum of a particle remains constant in a central field, i.e.

$$\mathbf{M} = [\mathbf{r}\mathbf{p}] = \text{const}$$

A vector product is perpendicular to the plane containing the vectors being multiplied. It follows that with a constant direction of the vector \mathbf{M} , the vector \mathbf{r} is always in one plane perpendicular to \mathbf{M} , and the trajectory of the particle is a plane curve. We shall determine the position of the particle with the aid of the polar coordinates r and φ , making the origin of coordinates coincide with the centre of the field. In these coordinates, the Lagrangian is [see formula (6.1)]

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - \frac{a}{r}$$

The function L does not contain the coordinate φ explicitly. Generalized coordinates not contained in a Lagrangian explicitly are called **cyclic**. In the absence of non-potential forces, Lagrange's equations corresponding to cyclic coordinates are as follows:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = 0$$

Hence,

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{const} \quad (11.2)$$

Therefore, the generalized momenta corresponding to the cyclic coordinates are constant, i.e. are integrals of motion.

In the problem we are considering, Eq. (11.2) has the form

$$p_{\varphi} = \frac{\partial L}{\partial \dot{\varphi}} = mr^2 \dot{\varphi} = M = \text{const} \quad (11.3)$$

We could have written this equation immediately, taking into account that $mr^2 \dot{\varphi}$ is the angular momentum of the particle relative to the origin of coordinates, which must be conserved in a central force field.

The energy of a particle in a central field is also an integral of motion. Consequently, calculations by formula (5.11) allow us to write

$$E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) + \frac{a}{r} = \text{const} \quad (11.4)$$

To find the trajectory of a particle, it is better to proceed from Eqs. (11.3) and (11.4) than from Lagrange's equations. This way is simpler because Lagrange's equations contain second derivatives of the coordinates, whereas Eqs. (11.3) and (11.4) contain the first derivatives of the coordinates with respect to time.

Excluding $\dot{\varphi}$ from Eqs. (11.3) and (11.4), we obtain

$$E = \frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} + \frac{a}{r}$$

whence

$$\dot{r} = \frac{dr}{dt} = \frac{1}{m} \sqrt{2mE - \frac{2am}{r} - \frac{M^2}{r^2}}$$

From Eq. (11.3)

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{1}{m} \frac{M}{r^2}$$

Excluding dt from the last two equations, we find that

$$d\varphi = \frac{(M/r^2) dr}{\sqrt{2mE - \frac{2am}{r} - \frac{M^2}{r^2}}} = \frac{(M/r^2) dr}{\sqrt{2mE + \left(\frac{am}{M}\right)^2 - \left(\frac{am}{M} + \frac{M}{r}\right)^2}}$$

Introducing the notation

$$2mE + \left(\frac{am}{M}\right)^2 = b^2, \quad \frac{am}{M} + \frac{M}{r} = u \quad \left(-\frac{M}{r^2} dr = du\right)$$

we can write that

$$\varphi = - \int \frac{du}{\sqrt{b^2 - u^2}} = \cos^{-1} \frac{u}{b} + \varphi_0$$

where φ_0 is an integration constant.

Returning to our previous notation, we get an equation of a particle's trajectory in polar coordinates:

$$\begin{aligned}\varphi - \varphi_0 &= \cos^{-1} \frac{am/M + M/r}{\sqrt{2mE + (am/M)^2}} \\ &= \cos^{-1} \frac{1 + (M^2/am)(1/r)}{\sqrt{1 + 2EM^2/ma^2}}\end{aligned}\quad (11.5)$$

Inspection of Eq. (11.5) shows that at a preset value of r , the difference $\varphi - \varphi_0$ can have two values differing in their sign [$\cos(-\alpha) = \cos \alpha$]. It is thus a simple matter to conclude that the curve described by Eq. (11.5) is symmetrical relative to the straight line making the angle φ_0 with the axis from which φ is measured.

To reveal the nature of the curve described by Eq. (11.5), let us introduce the notation

$$\frac{M^2}{|a|m} = p \quad (11.6)$$

$$\sqrt{1 + 2EM^2/ma^2} = e \quad (11.7)$$

The equation of the trajectory thus becomes

$$\varphi - \varphi_0 = \cos^{-1} \frac{1 \pm p/r}{e}$$

or, after simple transformations,

$$r = \mp \frac{p}{1 - e \cos(\varphi - \varphi_0)} \quad (11.8)$$

(the upper sign corresponds to repulsion, and the lower to attraction of the particle to the centre of force).

The equation we have obtained is one of a conic section (or conic—see Appendix IV) with the focal parameter p and the eccentricity e .

Let us first consider repulsion ($a > 0$). In this case, $U > 0$ so that the total energy E cannot be negative. Hence, by (11.7), we have $e > 1$. Consequently, in repulsion, the particle's trajectory can only be a hyperbola. Taking the upper (minus) sign in (11.8), we get the equation of the trajectory

$$r = \frac{-p}{1 - e \cos(\varphi - \varphi_0)}$$

The value of φ_0 is determined by the choice of the reference axis for φ . If the angle φ is measured from the axis of symmetry of the curve (from the straight line passing through its foci), r should not change when the sign of φ changes. This occurs only when $\varphi_0 = 0$ or $\varphi_0 = \pi$. Assuming that $\varphi_0 = 0$, we get the equation

$$r = \frac{-p}{1 - e \cos \varphi}$$

coinciding with Eq. (IV.14) that describes the right-hand branch of a hyperbola [provided that the origin of coordinates, i.e. the centre of force, is placed at the outer (left-hand) focus of the hyperbola].

Assuming that $\varphi_0 = \pi$ and taking into account that $\cos(\varphi - \pi) = -\cos \varphi$, we get the equation

$$r = \frac{-p}{1 + e \cos \varphi}$$

coinciding with Eq. (IV.13) that describes the left-hand branch of a hyperbola [provided that the origin of coordinates is placed at the outer (right-hand) focus of the hyperbola; Fig. 11.1].

Now let us consider attraction ($a < 0$). The lower (plus) sign in formula (11.8) corresponds to it. Hence, the equation of the trajectory is

$$r = \frac{p}{1 - e \cos \varphi} \quad (11.9)$$

for $\varphi_0 = 0$, and

$$r = \frac{p}{1 + e \cos \varphi} \quad (11.10)$$

for $\varphi_0 = \pi$.

As shown in Appendix IV, both equations describe either an ellipse, or one of the branches of a hyperbola, or a parabola [see Eqs. (IV.11) and (IV.12)]. The value of e determines which of these curves we have to do with.

In attraction, $U < 0$, consequently the total energy E may be either positive or negative, particularly, it may be zero. It follows from formula (11.7) that when $E > 0$, the eccentricity is greater than one, and the trajectory will be a hyperbola. Equation (11.9) gives the right-hand branch of a hyperbola, and Eq. (11.10), the left-hand one. Unlike repulsion, the origin of coordinates, i.e. the centre of force, is at the inner focus for the given branch (Fig. 11.2)¹.

At $E = 0$, the eccentricity $e = 1$, and the trajectory will be a parabola. This case occurs if a particle begins its motion from a state of rest at infinity.

¹ The solid curves in Fig. 11.2 are depicted for the same value of the angular momentum M and, consequently, for the same value of the focal parameter p . The dashed ellipse corresponds to a smaller value of M . For a smaller M , the vertex of a parabola may be to the right of that of a hyperbola corresponding to a greater M .

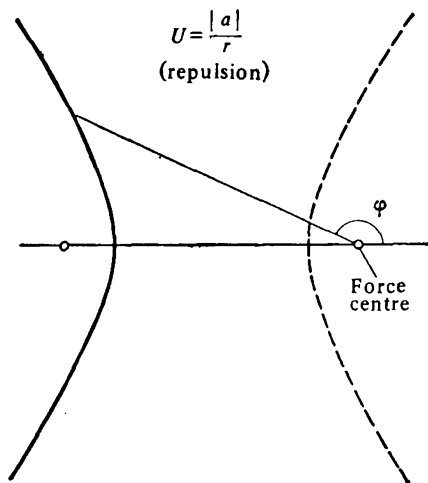


Fig. 11.1.

Finally, at $E < 0$, the eccentricity is less than one, and the trajectory will be an ellipse. In this case, the curves described by Eqs. (11.9) and (11.10) differ in the position of the centre of force. Curve (11.9) is obtained if the centre of force (the origin of coordinates)

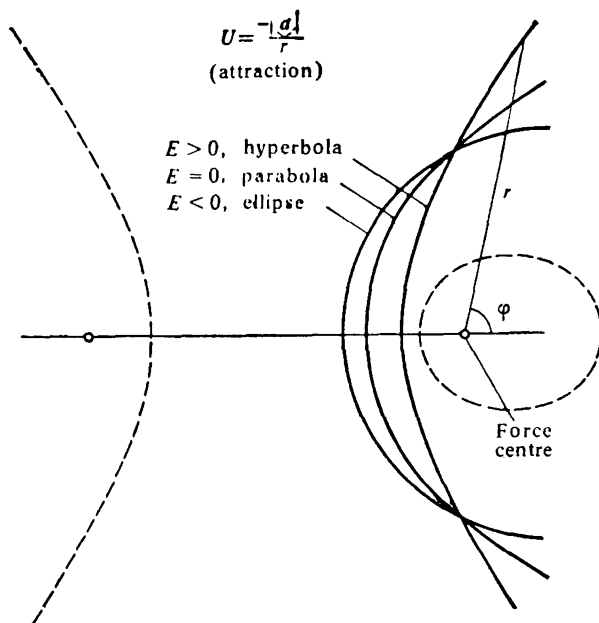


Fig. 11.2.

is at the left-hand focus of the ellipse. Curve (11.10) corresponds to the centre of force being at the right-hand focus.

If $1 + (2EM^2/ma^2) = 0$, i.e. $E = -ma^2/2M^2$, a glance at formula (11.7) shows that the eccentricity vanishes—the trajectory is a circle. At a given M , such an energy value in the given force field is the minimum possible one (an imaginary value of e corresponds to smaller E 's).

When a particle moves in a restricted region of space (the particle does not travel to infinity), the motion is called finite. In infinite motion, a particle travels to infinity. Motion in an ellipse is finite (recall that in this case $E < 0$), and motion in a hyperbola or parabola is infinite ($E \geq 0$).

12. Two-Body Problem

Consider a closed system formed of two interacting particles of masses m_1 and m_2 . The potential energy of the system depends on

the distance r between the particles. This distance can be treated as the magnitude of the vector \mathbf{r} drawn from one particle (say m_2) to the other one (m_1): $U = U(r)$ (Fig. 12.1).

The system has six degrees of freedom. We shall take as the generalized coordinates the three Cartesian coordinates of the centre of mass C of the system (which the position vector \mathbf{R} of the point C

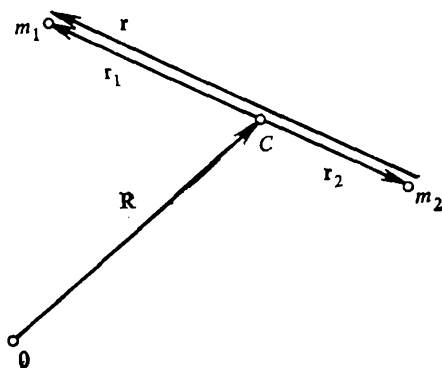


Fig. 12.1.

is equivalent to) and the three projections of the vector \mathbf{r} onto the coordinate axes (the vector \mathbf{r} is equivalent to them). The vector \mathbf{r} can be represented as

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \quad (12.1)$$

where \mathbf{r}_1 and \mathbf{r}_2 are the position vectors of the particles relative to the centre of mass. According to the definition of the centre of mass

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = 0 \quad (12.2)$$

Solving Eqs. (12.1) and (12.2) simultaneously, we find that

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2} \mathbf{r}, \quad \mathbf{r}_2 = -\frac{m_1}{m_1 + m_2} \mathbf{r} \quad (12.3)$$

The position of the particles relative to the origin of coordinates is determined by the vectors $\mathbf{R} + \mathbf{r}_1$ and $\mathbf{R} + \mathbf{r}_2$.

Let us write the Lagrangian of the system:

$$L = \frac{m_1}{2} (\dot{\mathbf{R}} + \dot{\mathbf{r}}_1)^2 + \frac{m_2}{2} (\dot{\mathbf{R}} + \dot{\mathbf{r}}_2)^2 - U(r)$$

Squaring and taking into account that by (12.2) we have $m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 = 0$, we obtain

$$L = \frac{m_1 + m_2}{2} \dot{\mathbf{R}}^2 + \frac{m_1}{2} \dot{\mathbf{r}}_1^2 + \frac{m_2}{2} \dot{\mathbf{r}}_2^2 - U(r)$$

Finally, substituting $\dot{\mathbf{r}}$ for $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ in accordance with (12.3), we arrive at an expression for L in the "coordinates" \mathbf{R} and \mathbf{r} which we have adopted¹:

$$L = \frac{m_1 + m_2}{2} \dot{\mathbf{R}}^2 + \frac{\mu}{2} \dot{\mathbf{r}}^2 - U(r) \quad (12.4)$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad (12.5)$$

is a quantity known as the reduced mass of the system.

Function (12.4) breaks up into two independent terms:

$$L = L(\dot{\mathbf{R}}) + L(\mathbf{r}, \dot{\mathbf{r}})$$

The first of them describes the behaviour of the centre of mass of the system, and the second, the motion of the particles relative to the centre of mass.

A glance at (12.4) shows that the coordinate \mathbf{R} is cyclic (it is not contained explicitly in L ; see Sec. 11). Consequently,

$$\mathbf{p}_R = (m_1 + m_2) \dot{\mathbf{R}} = \text{const}$$

(the momentum of the system is conserved; this could have been foreseen because the system is closed). The centre of mass of the system moves rectilinearly and uniformly (or is at rest).

The motion of particles relative to the centre of mass is described by the function

$$L = \frac{\mu}{2} \dot{\mathbf{r}}^2 - U(r) \quad (12.6)$$

It can be considered as the Lagrangian of a particle of mass μ moving in a central field with a stationary centre. The position of a particle relative to the force centre is determined by the position vector \mathbf{r} . Therefore, the problem on the motion of a system consisting of two interacting particles (the two-body problem) has been reduced to the problem on the motion of one particle in a central force field. We have treated this problem in Sec. 11. We established that when $U = a/r$, the trajectory of a particle will be a conic. Consequently, the tip of the position vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ slides in motion of the

¹ The projections of the vectors \mathbf{R} and \mathbf{r} onto the coordinate axes, and not the vectors themselves, are the actual coordinates.

particles along a curve that is a conic. By (12.3), the position vectors \mathbf{r}_1 and \mathbf{r}_2 are proportional to the position vector¹ \mathbf{r} . Therefore, each of these vectors also circumscribes a conic. Depending on the nature of the interaction (attraction or repulsion) and the magnitude of the total energy of the system, the trajectory of each of the particles will be either an ellipse, a parabola, or a hyperbola.

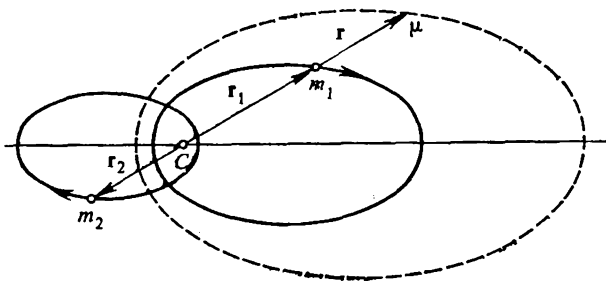


Fig. 12.2.

Assume that an auxiliary imaginary particle μ moves in an ellipse having the equation

$$r = \frac{p}{1 - e \cos \varphi}$$

According to (12.3), the vector \mathbf{r}_1 at any instant has the same direction as \mathbf{r} (i. e. $\varphi_1 = \varphi$), while its magnitude is $m_2/(m_1 + m_2)$ times greater. Hence, the particle m_1 moves in the ellipse

$$r_1 = \frac{p_1}{1 - e \cos \varphi_1} \quad (12.7)$$

where $p_1 = pm_2/(m_1 + m_2)$.

The vector \mathbf{r}_2 at each instant is directed oppositely to the vector \mathbf{r} [see (12.3)]. Consequently, when the vector \mathbf{r} is oriented at the angle φ , the vector \mathbf{r}_2 is oriented at the angle $\varphi_2 = \varphi + \pi$. The magnitude of the vector \mathbf{r}_2 is $m_1/(m_1 + m_2)$ times greater. Hence, having in view that $\cos(\varphi_2 - \pi) = -\cos \varphi_2$, the equation of the ellipse along which the particle m_2 travels must be written as

$$r_2 = \frac{p_2}{1 + e \cos \varphi_2} \quad (12.8)$$

where $p_2 = pm_1/(m_1 + m_2)$.

¹ For this reason, we must assume that the imaginary centre of force under whose action the particle μ moves is situated at the point from which the vectors \mathbf{r}_1 and \mathbf{r}_2 emerge, i.e. at the centre of mass of the system C .

In the case corresponding to Eq. (12.7), the origin of coordinates (i.e. the centre of mass C) is at the left-hand focus of the ellipse [see formula (IV.14)]. In the case corresponding to Eq. (12.8), the origin of coordinates (the point C) is at the right-hand focus of the ellipse [see formula (IV.12)]. Consequently, the trajectories of the particles are as shown in Fig. 12.2¹.

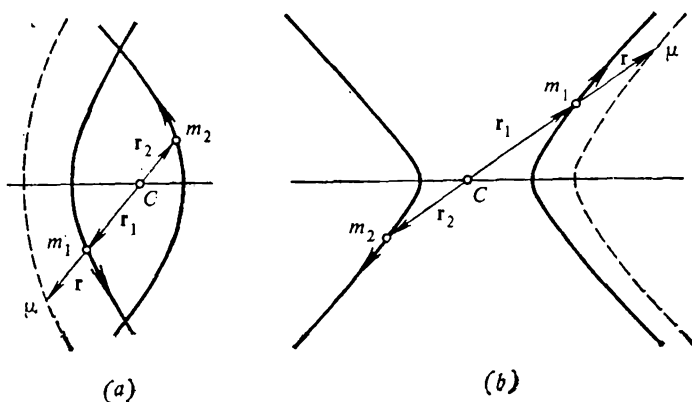


Fig. 12.3.

We invite our reader to see for himself that in motion along hyperbolas the trajectories of particles will appear as shown in Fig. 12.3a (for mutual attraction of the particles) and in Fig. 12.3b (for repulsion).

13. Elastic Collisions of Particles

A collision is defined to be a process consisting in that particles interacting with each other and arriving from infinity (i.e. from a distance such that their interaction may be disregarded) approach each other, and then either recede again to infinity, or remain at a finite distance from each other. In the first case, the collision is called the **scattering** of the particles, and in the second, their **capture**. The latter can obviously be observed only if the interaction of the particles has the nature of attraction.

When we speak about collisions of particles, we do not at all assume that the particles come into contact, as is the case, for instance, in the collision of two spheres. What we have in mind is only

¹ Figures 12.2 and 12.3 are for $m_1/m_2 = 2/3$; in Fig. 12.2, $e = 0.8$, and in Fig. 12.3, $e = 1.5$.

the fact that owing to their interaction, the particles change the direction or nature of their motion.

Consider the elastic collision of two particles repelling each other. A collision is defined as **elastic** if it is not attended by a change in the internal energy of the particles. Consequently, in elastic collisions, the mechanical energy of the system of colliding particles remains constant.

It is the simplest to consider a collision process in a reference frame associated with the centre of mass of the particles (it is known as a *c*-frame). In practice, however, collisions are observed in a reference frame relative to which the centre of mass of the particles moves with the velocity \mathbf{v}_C . This reference frame is called a laboratory one or, more briefly, an *l*-frame. In a laboratory frame, one of the particles is usually at rest before a collision.

The following relation obviously holds between the velocity of the *i*-th particle in an *l*-frame (we shall designate it by the symbol \mathbf{v}_i) and the velocity of the same particle in a *c*-frame [we shall designate this velocity by $\mathbf{v}_i^{(C)}$]:

$$\mathbf{v}_i = \mathbf{v}_C + \mathbf{v}_i^{(C)} \quad (13.1)$$

It follows from the definition of the centre of mass that

$$\mathbf{v}_C = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$$

In the following, we shall treat only the case when in an *l*-frame the second particle is at rest before a collision. Therefore, designating the velocity of the first particle before the collision by the symbol \mathbf{v}_{10} , we have

$$\mathbf{v}_C = \frac{m_1}{m_1 + m_2} \mathbf{v}_{10} \quad (13.2)$$

Using this value of \mathbf{v}_C in formula (13.1), we get the following expressions for the velocities of the particles in a *c*-frame before the collision:

$$\left. \begin{aligned} \mathbf{v}_{10}^{(C)} &= \frac{m_2}{m_1 + m_2} \mathbf{v}_{10} \\ \mathbf{v}_{20}^{(C)} &= -\frac{m_1}{m_1 + m_2} \mathbf{v}_{10} \end{aligned} \right\} \quad (13.3)$$

Multiplying the first of these velocities by m_1 and the second by m_2 , we find the momenta of the particles before the collision in the *c*-frame:

$$\begin{aligned} \mathbf{p}_{10}^{(C)} &= \mu \mathbf{v}_{10} \\ \mathbf{p}_{20}^{(C)} &= -\mu \mathbf{v}_{10} \end{aligned}$$

Here $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the particles

As expected, the total momentum of the particles in the c -frame before they collide is **zero**. It follows from the law of momentum conservation that after the collision too, the momenta of the particles in the c -frame can differ only in their sign: $\mathbf{p}_1^{(C)} = -\mathbf{p}_2^{(C)}$.

The total kinetic energy of the particles as a result of an elastic collision] cannot change (we assume that both before and after colliding the particles are so far from each other that their mutual potential energy is negligibly small). We can therefore write the relation

$$\frac{p_{10}^2}{2m_1} + \frac{p_{20}^2}{2m_2} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$$

[we have omitted the superscript "(C)" on the symbols of the momenta]. In combination with the condition that $|\mathbf{p}_{10}^{(C)}| = |\mathbf{p}_{20}^{(C)}|$ and $|\mathbf{p}_1^{(C)}| = |\mathbf{p}_2^{(C)}|$, the relation we have written indicates that the momenta (and, consequently, the velocities) of the particles as a result of colliding only turn in the c -frame through a certain angle χ , remaining constant in magnitude. Let \mathbf{e}_1 stand for the unit vector of the first particle's velocity in the c -frame after the collision. In accordance with formulas (13.3), we can therefore write the following expressions for the velocities of the particles after the collision:

$$\begin{aligned}\mathbf{v}_1^{(C)} &= \frac{m_2}{m_1 + m_2} v_{10} \mathbf{e}_1 \\ \mathbf{v}_2^{(C)} &= -\frac{m_1}{m_1 + m_2} v_{10} \mathbf{e}_1\end{aligned}$$

To obtain the velocities of the particles after the collision in the l -frame, we substitute expression (13.2) for \mathbf{v}_C and the above expressions for $\mathbf{v}_i^{(C)}$ in formula (13.1). The result is

$$\begin{aligned}\mathbf{v}_1 &= \frac{m_1}{m_1 + m_2} \mathbf{v}_{10} + \frac{m_2}{m_1 + m_2} v_{10} \mathbf{e}_1 \\ \mathbf{v}_2 &= \frac{m_1}{m_1 + m_2} \mathbf{v}_{10} - \frac{m_1}{m_1 + m_2} v_{10} \mathbf{e}_1\end{aligned}$$

We obtain the following expressions for the momenta of the particles in the l -frame after the collision:

$$\left. \begin{aligned}\mathbf{p}_1 &= \frac{m_1}{m_1 + m_2} \mathbf{p}_{10} + \frac{m_2}{m_1 + m_2} p_{10} \mathbf{e}_1 \\ \mathbf{p}_2 &= \frac{m_2}{m_1 + m_2} \mathbf{p}_{10} - \frac{m_1}{m_1 + m_2} p_{10} \mathbf{e}_1\end{aligned} \right\} \quad (13.4)$$

The following geometrical construction will be an excellent illustration of the obtained relations. Let us depict the vector \mathbf{p}_{10} by the segment AD (Fig. 13.1) and let the point O on it divide the length of the vector in the ratio $m_1 : m_2$. We draw a circle passing through the tip of the vector \mathbf{p}_{10} with the point O as its centre. The radius of

this circle is $p_{10}m_2/(m_1 + m_2)$. If $m_1 < m_2$, the point A will be inside the circle (Fig. 13.1a); if $m_1 > m_2$, the point A will be outside the circle (Fig. 13.1b); if $m_1 = m_2$, the point A will be on the circle (Fig. 13.1c). We lay off from the point O at the angle χ relative to p_{10} the unit vector e_1 of the direction in which we assume the first particle to be flying relative to the c -frame.

Hence, the segment OB will depict the vector $e_1 p_{10} m_2 / (m_1 + m_2)$, and in accordance with formulas (13.4) the segment AB will depict the vector p_1 , and the segment BD , the vector p_2 .

The angle θ_1 between the vectors p_1 and p_{10} is called the **scattering angle**. It characterizes the deviation of the first particle observed in the l -frame. The angle θ_2 between the vectors p_2 and p_{10} is called the **recoil angle**. The sum $\theta_1 + \theta_2$ is called the **divergence angle** of the particles after the collision. The angles θ_1 and θ_2 can be expressed in terms of χ —the deflection angle of the first particle in the c -frame. With a view to the length of the segment OB being $p_{10}m_2/(m_1 + m_2)$, we can write

$$\tan \theta_1 =$$

$$\frac{[p_{10}m_2/(m_1 + m_2)] \sin \chi}{p_{10}m_1/(m_1 + m_2) + [p_{10}m_2/(m_1 + m_2)] \cos \chi}$$

or

$$\tan \theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi} \quad (13.5)$$

From the isosceles triangle OBD , we get the relation

$$\theta_2 = \frac{\pi - \chi}{2} \quad (13.6)$$

Inspection of Fig. 13.1a shows that the lighter particle can diverge on the heavier one ($m_1 < m_2$) in any direction (the point B can

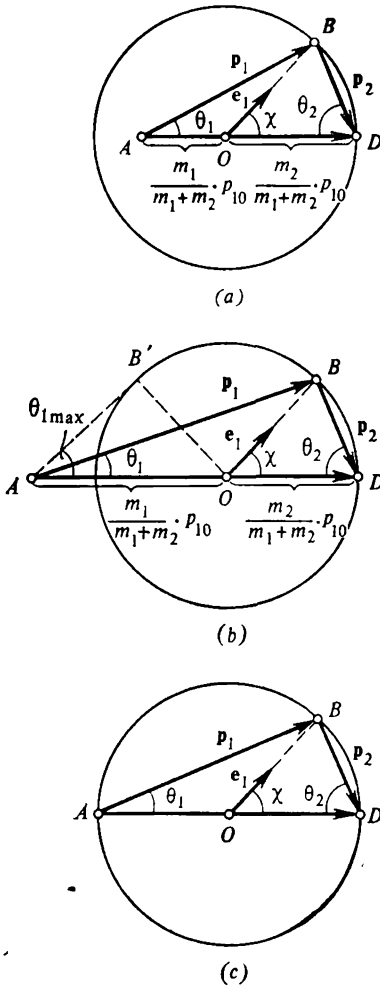


Fig. 13.1.

be at any place on the circle). The angle of divergence of the particles in this case is always greater than $\pi/2$.

When $m_1 > m_2$ (Fig. 13.1*b*), the scattering angle cannot exceed a certain extreme value $\theta_{1,\max}$ (the point B' in the figure corresponds to it). The sine of this angle equals the ratio of the segments OB' and OA , i.e.

$$\sin \theta_{1,\max} = \frac{m_2}{m_1}$$

When $m_1 > m_2$, the divergence angle of the particles is always less than $\pi/2$.

If the masses of the particles are the same ($m_1 = m_2$), the particles after colliding fly apart at right angles to each other ($\theta_1 + \theta_2 = \pi/2$; Fig. 13.1*c*).

In a head-on collision, the particles fly apart at the angle $\theta_1 + \theta_2$ equal either to π (when $m_1 < m_2$; Fig. 13.1*a*) or to zero (when $m_1 > m_2$; Fig. 13.1*b*). The angle χ in a head-on collision is π . When the masses of the particles are the same ($m_1 = m_2$), the momentum \mathbf{p}_1 is zero, and $\mathbf{p}_2 = \mathbf{p}_{10}$ (see Fig. 13.1*c*; in this case, the point B coincides with the point A). Consequently, particles of the same mass exchange momenta in a head-on collision. This result can also be obtained quite easily from formulas (13.4).

The results we have obtained are a corollary of the laws of energy and momentum conservation and do not depend on the nature of particle interaction. To determine at what angle χ a particle diverges, we must know the law of interaction of the particles and their mutual arrangement in colliding. The following section is devoted to a treatment of this matter.

14. Particle Scattering

We showed in Sec. 12 that the problem on the motion of two interacting particles reduces to the problem on the motion of a particle of mass μ (μ is the reduced mass) in a central force field, the distance from this particle to the centre of forces being equal to the distance between the particles in question. After finding the trajectory of the imaginary particle of mass μ , it is a simple matter to find the trajectories of both particles.

We shall use this procedure for studying the process of the divergence of the particle m_1 by the particle m_2 which is initially stationary in an l -frame. Let us pass over to a c -frame and consider the particle μ moving in a force field whose centre coincides with the centre of mass C of the system. We shall consider the field to be so weak at large distances from the centre that the motion of a particle at these distances may be considered rectilinear.

Let us introduce the **impact parameter** b equal to the distance from the centre of force at which a particle would fly past it if the field did not act on it (Fig. 14.1). It is evident that the angle of deflection of a particle is a function of the impact parameter: $\chi = \chi(b)$, and, generally speaking, χ should grow with diminishing b . Inversion of this function yields

$$b = b(\chi) \quad (14.1)$$

Assume that a beam of identical particles flying far from the centre of force C in the same direction and with the same velocity v_0

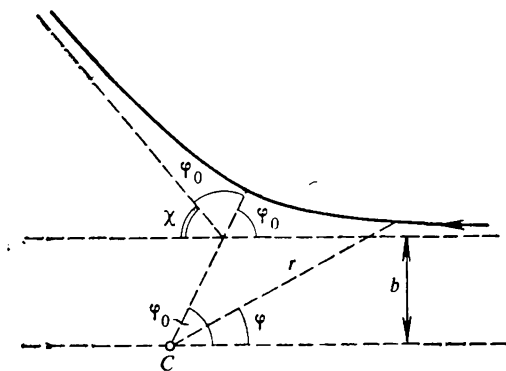


Fig. 14.1.

is incident on the centre. The beam can be characterized by the density of the particle flux j , by which is meant the number of particles flying a second through unit area at right angles to the beam. We shall assume that the beam of particles is homogeneous, i.e. that far from the scattering centre the flux density is the same at all points of the beam's cross section.

The beam particles deflect through different angles χ depending on the impact parameter of a particle approaching the centre. Particles whose impact parameter ranges from b to $b + db$ will be scattered within the limits of angles from χ to $\chi + d\chi$. Let us denote the flux of such particles (i.e. the number of particles scattered at angles from χ to $\chi + d\chi$ in unit time) by dN_χ . The ratio

$$d\sigma = \frac{dN_\chi}{j} \quad (14.2)$$

is known as the **differential effective cross section of scattering**. One reason why this name was introduced was the circumstance that $d\sigma$, as follows from (14.2), has the dimension of area. It is simple to see that $d\sigma$ determines the relative number (fraction) of particles scattered within a given range of angles.

When a beam has a homogeneous cross section, the flux of particles whose impact parameter ranges from b to $b + db$ is $j2\pi b db$ (the flux equals the flux density times the area). This flux is scattered at angles from χ to $\chi + d\chi$. Hence, $dN_\chi = j2\pi b db$. Using this value in formula (14.2), we obtain

$$d\sigma = \frac{j2\pi b db}{j} = 2\pi b db$$

or, passing over from the variable b to χ [see (14.1)],

$$d\sigma = 2\pi b(\chi) \left| \frac{db}{d\chi} \right| d\chi \quad (14.3)$$

(we have taken the absolute value of $db/d\chi$ because $db/d\chi$ is less than zero).

The quantity dN_χ in formula (14.2) can be interpreted as the flux of particles flying within the solid angle $d\Omega = 2\pi \sin \chi d\chi$ (this is the value of the solid angle confined between cones with apex angles of χ and $\chi + d\chi$). Substituting $d\Omega/\sin \chi$ for $2\pi d\chi$ in (14.3), we can reduce the formula for the differential effective cross section of scattering to the form

$$d\sigma = \frac{b(\chi)}{\sin \chi} \left| \frac{db}{d\chi} \right| d\Omega \quad (14.4)$$

Formula (14.3), like (14.4), is the most general—it determines the differential effective scattering cross section for any central scattering field. The quantities $b(\chi)$ and $db/d\chi$ are determined by the nature of the force field, i.e. by that of particle interaction. Consequently, $d\sigma$ is determined by the kind of scattering field and is the most important characteristic of a scattering process. We can procure information on the nature of a force field by experimentally studying $d\sigma$.

Up to now, we have dealt with the scattering of a beam of particles on one scattering centre. In practice, however, scattering occurs on a collection of identical scattering centres. In this connection, we shall note the following circumstance. The deflection angles χ are appreciable only for particles that approach the scattering centre sufficiently closely (for which the impact parameter is small). Therefore, if there are n identical, not overlapping, and sufficiently rarefied scattering centres in the path of particles, they will scatter the particles independently of one another, and the flux of particles deflected within the range of angles from χ to $\chi + d\chi$ will be n times greater than when there is only one centre. Hence, with n centres, we have

$$dN_\chi = nj d\sigma \quad (14.5)$$

Let us now go over from a particle of mass μ deflected by a stationary force centre at the point C to real particles m_1 and m_2 . The

trajectories of these particles are geometrically similar to that of the particle μ . Indeed, according to (12.3), the position vector of the first particle emerging from the point C is $m_2/(m_1 + m_2)$ times longer than the position vector of the particle μ emerging from the same point. A similar relation also holds for the second particle. This signifies that the particle m_1 deflects through the same angle χ as the particle μ in a c -frame. The impact parameter must be taken equal to the distance at which the first particle would fly past the second one if the particles did not interact (the position vector \mathbf{r} of the particle μ emerging from the force centre C coincides with $\mathbf{r}_1 - \mathbf{r}_2$).

Hence, formula (14.3) also holds (in a c -system) for a beam of particles m_1 scattered by the particle m_2 . To go over from the c -frame to the laboratory reference frame in which scattering is being observed, we must transfer from the variable χ to the variable θ_1 in formula (14.3). We use formula (13.5) for this transition. The resulting formula is very cumbersome in the general case.

In the particular case when $m_1 \ll m_2$ (the particles being scattered are much lighter than the ones scattering them), $\theta_1 \approx \chi$ [see formula (13.5)] so that formulas (14.3) and (14.4) can be written in the l -frame:

$$\left. \begin{aligned} d\sigma &= 2\pi b(\theta_1) \left| \frac{db}{d\theta_1} \right| d\theta_1 \\ d\sigma &= \frac{b(\theta_1)}{\sin \theta_1} \left| \frac{db}{d\theta_1} \right| d\Omega \end{aligned} \right\} \quad (14.6)$$

We must note that in this case $\mu \approx m_1$ and $\mathbf{r} \approx \mathbf{r}_1$ (the trajectory of the particle m_1 virtually coincides with that of the particle μ).

Consider a Coulomb scattering field, i.e. a field of the kind $U = a/r$, assuming that $m_1 \ll m_2$. The energy of the particle being deflected can be represented by the expression $E = \frac{1}{2} m_1 v_0^2$, where v_0 is the initial (and final) velocity of the particle m_1 . The angular momentum of the particle m_1 relative to the scattering centre (coinciding with the particle m_2) is $M = m_1 v_0 b$ (see Fig. 14.1). Using these values of E and M in formula (11.5), we arrive at the relation

$$\varphi - \varphi_0 = \cos^{-1} \frac{1 + (m_1 v_0^2 b^2 / a) (1/r)}{\sqrt{1 + (m_1 v_0^2 b / a)^2}} \quad (14.7)$$

A glance at Fig. 14.1 shows that when $r = \infty$ there are two values of φ —zero and $2\varphi_0$. In the first case, the left-hand side of formula (14.7) becomes $-\varphi_0$, and in the second, $+\varphi_0$. Therefore, assuming in (14.7) that $r = \infty$, we can write

$$\varphi_0 = \left| \cos^{-1} \frac{1}{\sqrt{1 + (m_1 v_0^2 b / a)^2}} \right|$$

whence

$$\cos^2 \varphi_0 = \frac{1}{1 + (m_1 v_0^2 b / a)^2} \quad (14.8)$$

Further, it follows from Fig. 14.1 that $\chi = \pi - 2\varphi_0$, i.e. $\varphi_0 = \pi/2 - \chi/2 = \pi/2 - \theta_1/2$ (in the case being considered it is possible to assume that $\theta_1 = \chi$). Introducing this value of φ_0 into formula (14.8), we obtain

$$\sin^2 \frac{\theta_1}{2} = \frac{1}{1 + (m_1 v_0^2 b/a)^2}$$

Solving this relation for b , we arrive after simple transformations at the expression

$$b = b(\theta_1) = \frac{a}{m_1 v_0^2} \cot \frac{\theta_1}{2} \quad (14.9)$$

Differentiation with respect to θ_1 yields

$$\frac{db}{d\theta_1} = -\frac{a}{m_1 v_0^2} \frac{1}{2 \sin^2 (\theta_1/2)} \quad (14.10)$$

Finally, using expressions (14.9) and (14.10) in (14.6), we get formulas for the differential effective cross section of scattering of the particles of mass m_1 in the Coulomb field set up by the particle of mass m_2 (here $m_2 \gg m_1$):

$$d\sigma = \pi \left(\frac{a}{m_1 v_0^2} \right)^2 \frac{\cos (\theta_1/2)}{\sin^3 (\theta_1/2)} d\theta_1 \quad (14.11)$$

$$d\sigma = \left(\frac{a}{2m_1 v_0^2} \right)^2 \frac{d\Omega}{\sin^4 (\theta_1/2)} \quad (14.12)$$

We have obtained Rutherford's formula for the scattering of alpha-particles on heavy nuclei known from the general course of physics. This can be verified by substituting $2Ze^2$ for a and multiplying formulas (14.11) and (14.12) by the flux density i of alpha-particles and the number of atoms n of the scattering substance per unit cross-sectional area of the alpha particle beam. The expression $n i d\sigma$ will be obtained on the left-hand side of the formula, which gives dN_{θ_1} —the flux of alpha particles scattered in the range of angles from θ_1 to $\theta_1 + d\theta_1$, or dN_{Ω} —the flux of alpha particles scattered in the solid angle $d\Omega$ [see formula (14.5)].

We must note that the expressions we have found for $d\sigma$ do not depend on the sign of a so that the result obtained holds not only for the Coulombian repulsion of the particles m_1 and m_2 , but also for their Coulombian attraction.

15. Motion in Non-Inertial Reference Frames

The Lagrangian of one particle has the form

$$L = \frac{1}{2} m v^2 - U(\mathbf{r}) \quad (15.1)$$

only in inertial reference frames. Let us find the form of L in an arbitrary non-inertial reference frame. Figure 15.1 depicts the iner-

tial reference frame K and the frame K' whose origin (the point O') moves in the frame K at the velocity $\mathbf{v}_0(t)$. The frame K' , in addition, rotates relative to the frame K at the angular velocity $\omega(t)$. Let us express the function (15.1) in terms of the vector \mathbf{r}' determining the position of a particle in the frame K' , and in terms of the velocity \mathbf{v}' of the particle observed in the same frame.

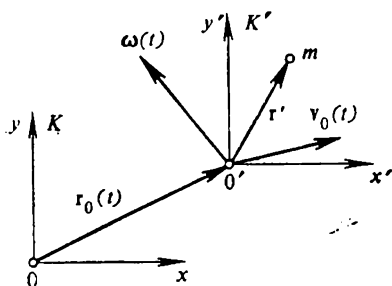


Fig. 15.1.

We assume first that $\mathbf{v}_0(t) \equiv 0$ and that the origins of both reference frames coincide. Hence, the following relation would hold between the velocities of the particle in both frames:

$$\mathbf{v} = \mathbf{v}' + [\omega \mathbf{r}'] \quad (15.2)$$

(a particle that is stationary in the frame K' would have a velocity equal to $[\omega \mathbf{r}']^1$ in the frame K). If $\mathbf{v}_0(t)$ is non-zero, on the other hand, relation (15.2) becomes

$$\mathbf{v} = \mathbf{v}_0(t) + \mathbf{v}' + [\omega \mathbf{r}'] \quad (15.3)$$

Let us introduce the expression we have obtained for \mathbf{v} into formula (15.1). This yields

$$L = \frac{m}{2} [\mathbf{v}_0(t)]^2 + \frac{m\mathbf{v}'^2}{2} + \frac{m}{2} [\omega \mathbf{r}']^2 + m\mathbf{v}_0(t) \mathbf{v}' + m\mathbf{v}_0(t) [\omega \mathbf{r}'] + m\mathbf{v}' [\omega \mathbf{r}'] \quad (15.4)$$

The first term in this formula is the preset function of time, which can be represented as the total derivative with respect to t of another function. We established in Sec. 7 that the Lagrangian must be determined to within the additive terms that are the total time derivative of an arbitrary function of the generalized coordinates and time. For this reason, the term $(m/2) [\mathbf{v}_0(t)]^2$ should be omitted.

Consider the fourth and fifth terms in formula (15.4). Factoring out $m\mathbf{v}_0(t)$, these terms can be written in the form

$$m\mathbf{v}_0(t) \{ \mathbf{v}' + [\omega \mathbf{r}'] \} = m\mathbf{v}_0(t) \left\{ \frac{d'\mathbf{r}'}{dt} + \left[\frac{d\varphi}{dt}, \mathbf{r}' \right] \right\} = m\mathbf{v}_0(t) \frac{\{d'\mathbf{r}' + [d\varphi, \mathbf{r}']\}}{dt} \quad (15.5)$$

Here $d'\mathbf{r}'$ is the increment of \mathbf{r}' observed during the time dt in the frame K' (we remind our reader that \mathbf{v}' is the velocity of the particle

¹ This expression is obtained from formula (VI.46) if we assume that $\mathbf{a} = \mathbf{r}'$ in it and divide the relation obtained by dt .

observed in the frame K'), and $d\varphi$ is the angle through which the frame K' turns during the time dt .

If a reference frame rotates relative to another one, the increment of a vector \mathbf{a} observed in both frames will be different. This is easy to understand by assuming that the vector does not change with respect to the rotating frame, i.e. the increment of the vector in this frame (we designate it by K') is zero: $d'\mathbf{a} = 0$. Consequently, the increment of the vector in the stationary frame (the frame K) can be written as

$$d\mathbf{a} = [d\varphi, \mathbf{a}]$$

[see formula (VI.46)]. If the increment of the vector $d'\mathbf{a}$ observed in the rotating frame is non-zero, the increment observed in the stationary frame will be

$$d\mathbf{a} = d'\mathbf{a} + [d\varphi, \mathbf{a}] \quad (15.6)$$

[assuming that $\mathbf{a} = \mathbf{r}'$ and dividing by dt , we arrive at formula (15.2)].

By comparing (15.6) with the expression in braces on the right-hand side of formula (15.5), we arrive at the conclusion that this expression is the increment of the vector \mathbf{r}' observed in the frame K , i.e. $d\mathbf{r}'$. Hence, the sum of the fourth and fifth terms of formula (15.4) can be written as

$$m\mathbf{v}_0(t) \frac{d\mathbf{r}'}{dt}$$

Let us transform this expression as follows:

$$m\mathbf{v}_0(t) \frac{d\mathbf{r}'}{dt} = \frac{d}{dt} \{m\mathbf{v}_0(t) \mathbf{r}'\} - m\mathbf{r}' \frac{d\mathbf{v}_0}{dt}$$

We may discard the first term as the total time derivative of the function of the coordinates and time. In the second term, $d\mathbf{v}_0/dt$ is $\mathbf{w}_0(t)$ —the acceleration of the origin of coordinates of the frame K' observed in the frame K .

We have thus arrived at the following expression for the Lagrangian in the variables \mathbf{r}' and \mathbf{v}' :

$$L' = \frac{1}{2} m \mathbf{v}'^2 + \frac{1}{2} m [\omega \mathbf{r}']^2 - m \mathbf{r}' \mathbf{w}_0(t) + m \mathbf{v}' [\omega \mathbf{r}'] - U(\mathbf{r}') \quad (15.7)$$

We have obtained the general form for the Lagrangian of a particle in an arbitrary non-inertial reference frame. We must now consider that the function U is set in the variables \mathbf{r}' [in formula (15.1) it was set in the variables \mathbf{r}]. The transition from one set of variables to another is accomplished by the formula

$$\mathbf{r} = \mathbf{r}_0(t) + \mathbf{r}' \quad (15.8)$$

where $\mathbf{r}_0(t)$ is the position vector of the origin of coordinates of the frame K' (see Fig. 15.1).

We must note that even if the time were not contained explicitly in the function (15.1) (it could have been contained in the force function U), the function (15.7) contains the time because \mathbf{w}_0 and $\boldsymbol{\omega}$ are, generally speaking, functions of t . The time also enters explicitly the term $U(\mathbf{r}')$ as a result of the transition from \mathbf{r} to \mathbf{r}' accomplished by formula (15.8).

Before beginning to compile Lagrange's equation, let us replace the second term of expression (15.7) in accordance with formula (VI.6). The result is

$$L' = \frac{1}{2} m \mathbf{v}'^2 + \frac{1}{2} m \omega^2 \mathbf{r}'^2 - \frac{1}{2} m (\boldsymbol{\omega} \mathbf{r}')^2 - m \mathbf{r}' \mathbf{w}_0(t) + m \mathbf{v}' [\boldsymbol{\omega} \mathbf{r}'] - U(\mathbf{r}') \quad (15.9)$$

Taking advantage of a cyclic transposition of the multipliers [see formula (VI.3)], the next-to-last term equal to $m \mathbf{v}' [\boldsymbol{\omega} \mathbf{r}']$ could be written in the form

$$m [\mathbf{v}' \boldsymbol{\omega}] \mathbf{r}' \quad (15.10)$$

Lagrange's equation in the frame K' is as follows:

$$\frac{d}{dt} \frac{\partial L'}{\partial \mathbf{v}'} = \frac{\partial L'}{\partial \mathbf{r}'} \quad (15.11)$$

[see formula (9.3) and the footnote on page 17]. A glance at expression (15.9) shows that

$$\frac{\partial L'}{\partial \mathbf{v}'} = m \mathbf{v}' + m [\boldsymbol{\omega} \mathbf{r}']$$

whence

$$\frac{d}{dt} \frac{\partial L'}{\partial \mathbf{v}'} = m \dot{\mathbf{v}}' + m [\dot{\boldsymbol{\omega}} \mathbf{r}'] + m [\boldsymbol{\omega} \dot{\mathbf{r}}']$$

We remind our reader that from the very instant when we expressed L in the variables \mathbf{r}' and \mathbf{v}' , we have been "living" in the reference frame K' . Consequently, by $\dot{\mathbf{v}}'$ we must understand the acceleration \mathbf{w}' of a particle observed in the frame K' , and by $\dot{\mathbf{r}}'$, the velocity \mathbf{v}' of the particle in the same frame. Hence,

$$\frac{d}{dt} \frac{\partial L'}{\partial \mathbf{v}'} = m \mathbf{w}' + m [\dot{\boldsymbol{\omega}} \mathbf{r}'] + m [\boldsymbol{\omega} \mathbf{v}'] \quad (15.12)$$

As regards $\dot{\boldsymbol{\omega}}$, it is the time derivative of the function $\boldsymbol{\omega}(t)$ that is set in the frame K .

In calculating $\partial L' / \partial \mathbf{r}'$, we shall assume that the next-to-last term in formula (15.9) is represented in the form (15.10). We thus obtain

$$\frac{\partial L'}{\partial \mathbf{r}'} = m \omega^2 \mathbf{r}' - m (\boldsymbol{\omega} \mathbf{r}') \boldsymbol{\omega} - m \mathbf{w}_0(t) + m [\mathbf{v}' \boldsymbol{\omega}] - \frac{\partial U}{\partial \mathbf{r}'} \quad (15.13)$$

The first two terms in this expression are the triple vector product $m[\omega, [\mathbf{r}'\dot{\omega}]]$ written according to formula (VI.5). Expression (15.13) can therefore be written as

$$\frac{\partial L'}{\partial \mathbf{r}'} = m[\omega, [\mathbf{r}'\dot{\omega}]] - m\mathbf{w}_0(t) + m[\mathbf{v}'\dot{\omega}] - \frac{\partial U}{\partial \mathbf{r}'} \quad (15.14)$$

Introducing expressions (15.12) and (15.14) into formula (15.11) and performing transformations, we arrive at an equation of motion of a particle in the frame K' :

$$m\mathbf{w}' = -\frac{\partial U}{\partial \mathbf{r}'} - m\mathbf{w}_0(t) + m[\mathbf{r}'\dot{\omega}] + m[\omega, [\mathbf{r}'\dot{\omega}]] + 2m[\mathbf{v}'\dot{\omega}] \quad (15.15)$$

We see that the acceleration of a particle in the frame K' is determined, in addition to the force $-\partial U/\partial \mathbf{r}'$ due to the force field, by a number of additional forces called, as is well known, forces of inertia. The term $m[\omega, [\mathbf{r}'\dot{\omega}]]$ gives the centrifugal force of inertia, and the term $2m[\mathbf{v}'\dot{\omega}]$, the Coriolis force. The force $m[\mathbf{r}'\dot{\omega}]$ is associated with the non-uniformity of rotation; it has no special name.

If the frame K' has only translational motion relative to the frame K (in this case $\omega = 0$ and, consequently, $\dot{\omega}$ also equals 0), the equation of motion contains only one force of inertia equal to

$$\mathbf{f}_{\text{in}} = -m\mathbf{w}_0(t) \quad (15.16)$$

It is remarkable that this force, like the force of gravity $m\mathbf{g}$, is proportional to the mass of the particle. This circumstance underlies the general theory of relativity.

For a uniformly rotating system of coordinates having no translational acceleration [$\mathbf{w}_0(t) = 0$, $\dot{\omega} = 0$], the Lagrangian has the form [see (15.7)]

$$L' = \frac{m\mathbf{v}'^2}{2} + \frac{m[\omega\mathbf{r}']^2}{2} + m\mathbf{v}'[\omega\mathbf{r}'] - U \quad (15.17)$$

Let us find the momentum, the angular momentum, and the energy of a particle for this case. By formula (9.5)

$$\mathbf{p}' = \frac{\partial L'}{\partial \mathbf{v}'}$$

Taking the derivative of function (15.17) with respect to \mathbf{v}' , we get

$$\mathbf{p}' = m\mathbf{v}' + m[\omega\mathbf{r}'] = m\{\mathbf{v}' + [\omega\mathbf{r}']\} \quad (15.18)$$

If the frame K' has neither a translational acceleration nor a translational velocity ($\mathbf{v}_0 = 0$), inspection of (15.3) shows that the expression in braces in (15.18) is the velocity \mathbf{v} of a particle relative to the inertial frame K . Hence, \mathbf{p}' equals $m\mathbf{v}$, i.e. it coincides with the momentum \mathbf{p} of a particle in the inertial frame:

$$\mathbf{p}' = \mathbf{p} \quad (15.19)$$

Further, if the origins of the frames K and K' coincide [see Fig. 15.1)], the position vectors \mathbf{r} and \mathbf{r}' also coincide. Hence, with a view to (15.19), it follows that the angular momentum $\mathbf{M}' = [\mathbf{r}'\mathbf{p}']$ in the frame K' coincides with the angular momentum $\mathbf{M} = [\mathbf{r}\mathbf{p}]$ in the frame K :

$$\mathbf{M}' = \mathbf{M} \quad (15.20)$$

By formula (5.1), the energy of a particle in the frame K' is determined by the expression

$$E' = \sum_i \frac{\partial L'}{\partial \dot{x}_i'} \dot{x}_i' - L'$$

where x_i' are the Cartesian coordinates of the particle in the frame K' . According to (4.17), $\partial L'/\partial \dot{x}_i'$ is p_i' —the projection of the momentum \mathbf{p}' of the particle onto the i -th coordinate axis, and \dot{x}_i' is the projection of the velocity \mathbf{v}' of the particle onto the same axis. Consequently, the expression for the energy can be written as

$$E' = \mathbf{p}'\mathbf{v}' - L' \quad (15.21)$$

Substituting for \mathbf{p}' its value from (15.18) and for L' expression (15.17), we get the following formula:

$$E' = \frac{m\mathbf{v}'^2}{2} + U - \frac{m}{2} [\boldsymbol{\omega}\mathbf{r}']^2 \quad (15.22)$$

Rotation of the reference frame manifested itself in the appearance of the term

$$U_{cf} = -\frac{m}{2} [\boldsymbol{\omega}\mathbf{r}']^2 \quad (15.23)$$

not depending on the particle's velocity v' in the expression for the energy. This additional "potential" energy is called centrifugal.

Let us substitute $\mathbf{v} - [\boldsymbol{\omega}\mathbf{r}]$ for \mathbf{v}' in formula (15.22) [see (15.3)]; we assume that $\mathbf{v}_0(t) = 0$. The result is

$$E' = \frac{m\mathbf{v}^2}{2} + U - m\mathbf{v} [\boldsymbol{\omega}\mathbf{r}] \quad (15.24)$$

The first two terms give the energy E of a particle in the frame K . If the origins of the frames K and K' coincide, \mathbf{r}' may be replaced by \mathbf{r} . The last term in (15.24) by means of cyclic transposition can now be given the form

$$m\mathbf{v} [\boldsymbol{\omega}\mathbf{r}] = \boldsymbol{\omega} [\mathbf{r}, m\mathbf{v}] = \boldsymbol{\omega}\mathbf{M}$$

Therefore, the following relation holds between the energies E and E' of the particle in the frames K and K' respectively:

$$E' = E - \boldsymbol{\omega}\mathbf{M} \quad (15.25)$$

We remind our reader that this formula has been obtained assuming that the origins of both reference frames coincide. Consequently, instead of \mathbf{M} in formula (15.25), we can write \mathbf{M}' [see (15.20)].

Summarizing, if the reference frame K' rotates uniformly relative to the reference frame K , and the origins of both frames coincide, the momentum and the angular momentum of a particle in both frames coincide, while the energy of the particle in the frame K' is less than that in the frame K by the magnitude of the scalar product of the vectors $\boldsymbol{\omega}$ and \mathbf{M} .

Chapter IV

SMALL-AMPLITUDE OSCILLATIONS

16. Free Oscillations of a System Without Friction

Consider a system with one degree of freedom in which friction forces are absent. The potential energy of such a system has the form $U = U(q)$, where q is a generalized coordinate. The potential energy is known to be minimum in the position of stable equilibrium. We shall measure q from this position. Let us expand the function $U(q)$ in powers of q in close proximity to the point $q = 0$. Owing to the smallness of q , we shall limit ourselves to the first terms of the expansion:

$$U(q) = U(0) + U'(0)q + \frac{1}{2}U''(0)q^2$$

The condition of equilibrium yields $U'(0) = 0$. Let us measure the potential energy from the equilibrium position, i.e. assume that $U(0) = 0$. Finally, let us introduce the symbol $U''(0) = \kappa$ (remember that the second derivative is positive at a point of a minimum, hence, $\kappa > 0$). As a result, we arrive at the expression

$$U(q) = \frac{\kappa q^2}{2} \quad (16.1)$$

We shall consider the constraints to be stationary. Therefore, by (5.7)

$$T = \gamma(q) \dot{q}^2$$

In passing through the equilibrium position, T does not vanish. Consequently, $\gamma(0)$ is non-zero. Expanding $\gamma(q)$ into a series and retaining only the zero term of the expansion owing to the smallness of q , we can write

$$T = \frac{\mu \dot{q}^2}{2} \quad (16.2)$$

where $\mu = 2\gamma(0)$ (do not confuse it with the reduced mass!).

Let us compile the Lagrangian:

$$L = \frac{\mu \dot{q}^2}{2} - \frac{\kappa q^2}{2} \quad (16.3)$$

Lagrange's equation

$$\ddot{mq} + \kappa q = 0 \text{ or } \ddot{q} + \omega_0^2 q = 0 \quad (16.4)$$

[here $\omega_0^2 = (\kappa/\mu) > 0$] is a linear homogeneous second-order differential equation with constant coefficients (see Appendix V). Using the substitution $q = e^{\lambda t}$, we arrive at the characteristic equation¹

$$\lambda^2 + \omega_0^2 = 0$$

The roots of this equation are $\lambda_1 = +i\omega_0$ and $\lambda_2 = -i\omega_0$. Consequently, the general solution has the form

$$q = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} \quad (16.5)$$

where C_1 and C_2 are complex constants.

The values of q must be real; this signifies that the condition $q^* = \overline{q}$ (q^* is the complex conjugate of q) must be observed. Introducing expression (16.5) for q into this condition, we obtain

$$C_1^* e^{-i\omega_0 t} + C_2^* e^{i\omega_0 t} = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$$

The above relation is observed if $C_1 = C_2^*$ (correspondingly $C_1^* = C_2$). Having this in view, we shall write the coefficients C_1 and C_2 as

$$C_1 = \frac{a}{2} e^{i\alpha}, \quad C_2 = \frac{a}{2} e^{-i\alpha} \quad (16.6)$$

(a and α are arbitrary real constants). The use of these values in formula (16.5) yields

$$q = \frac{a}{2} \{e^{i(\omega_0 t + \alpha)} + e^{-i(\omega_0 t + \alpha)}\} = a \cos(\omega_0 t + \alpha) \quad (16.7)$$

Therefore, the free motion of the system near the position of stable equilibrium has the nature of a harmonic oscillation (naturally, provided that q remains small in the process of motion).

It is known from the general course of physics that a is called the amplitude, α —the initial phase of the oscillation, and ω_0 —the natural frequency of the system².

Let us transform expression (16.7) according to the formula for the cosine of a sum:

$$q = a (\cos \alpha \cos \omega_0 t - \sin \alpha \sin \omega_0 t)$$

and introduce the notation

$$c_1 = a \cos \alpha, \quad c_2 = -a \sin \alpha$$

The solution of Eq. (16.4) can thus be written as

$$q = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t \quad (16.8)$$

¹ See formulas (V.7) and (V.9).

² As a rule, we shall not give the information on a question being considered that can be found in textbooks of general physics.

where c_1 and c_2 are real constants whose values are determined from the initial conditions [from q_0 and $(\dot{q})_0$].

Finally, we shall give another form of writing a harmonic oscillation¹:

$$q = \operatorname{Re} \{\hat{q}\} = \operatorname{Re} \{\hat{A}e^{i\omega_0 t}\} \quad (16.9)$$

where

$$\hat{A} = ae^{i\alpha} \quad (16.10)$$

is a complex amplitude; its magnitude equals the ordinary amplitude, and its argument equals the initial phase of the oscillation. Introducing the value of \hat{A} from (16.10) into (16.9) and taking the real part of the expression obtained, we arrive at formula (16.7).

Consequently, a harmonic oscillation can be represented in the form of any of the three formulas (16.7), (16.8), or (16.9).

17. Damped Oscillations

In a real oscillatory system, forces act that retard the motion of the system and lead to a gradual attenuation of the amplitudes (damping) of the oscillations. The mechanical energy of the system transforms into the internal energy of the system and the surroundings (for brevity, the energy is usually said to transform into heat, but this is not quite strict). Such a process is called the dissipation of energy.

We shall limit ourselves to a treatment of cases when the generalized force of friction retarding a system is proportional to the generalized velocity of the system:

$$Q^* = -r\dot{q}$$

This is a non-potential force, therefore Lagrange's equation will have the form of (4.15), and the function (16.3) must be taken as L . Hence, damped oscillations are described by the equation

$$\mu\ddot{q} + \kappa\dot{q} = -r\dot{q}$$

(the symbols μ and κ have the same meaning as in the preceding section). Let us write this equation in the form

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = 0 \quad (17.1)$$

where

$$\omega_0^2 = \frac{\kappa}{\mu} > 0 \text{ and } 2\beta = \frac{r}{\mu}$$

The substitution $q = e^{\lambda t}$ leads to the characteristic equation

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0 \quad (17.2)$$

¹ We shall use a cap over a symbol to designate complex quantities.

Provided that $\beta^2 < \omega_0^2$, the roots of the characteristic equation are complex:

$$\lambda_1 = -\beta + i\sqrt{\omega_0^2 - \beta^2}, \quad \lambda_2 = -\beta - i\sqrt{\omega_0^2 - \beta^2}$$

The general solution of Eq. (17.1) is

$$q = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = e^{-\beta t} (C_1 e^{i\omega t} + C_2 e^{-i\omega t})$$

where $\omega = \sqrt{\omega_0^2 - \beta^2}$. The solution we have found differs from the function (16.5) in the factor $e^{-\beta t}$ and in the substitution of ω for ω_0 .

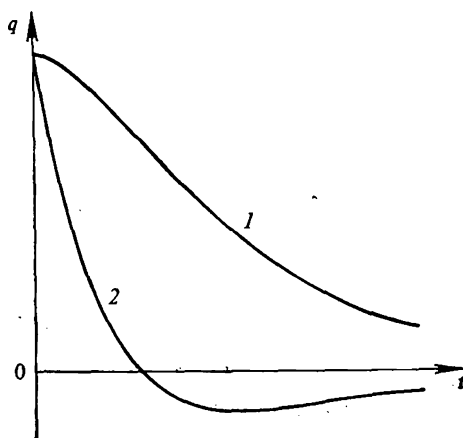


Fig. 17.1.

The requirement that q be real leads to the condition $C_1 = C_2^*$. Introducing the notation (16.6) and performing elementary transformations, we arrive at an expression for damped oscillations:

$$q = ae^{-\beta t} \cos(\omega t + \alpha) \quad (17.3)$$

When $\beta^2 > \omega_0^2$, the roots of characteristic equation (17.2) are real:

$$\lambda_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} = -\alpha_1, \quad \lambda_2 = -\beta - \sqrt{\beta^2 - \omega_0^2} = -\alpha_2$$

(since $\sqrt{\beta^2 - \omega_0^2} < \beta$, the quantity α_1 is positive; the quantity α_2 is also positive, and $\alpha_2 > \alpha_1$). The solution in this case is

$$q = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t} \quad (17.4)$$

where C_1 and C_2 are real constants.

Hence, with strong friction (when $\beta^2 > \omega_0^2$), no oscillations appear—the system brought out from its equilibrium position returns to it asymptotically. The motion of the system may have the nature described either by curve 1 or curve 2 (Fig. 17.1). In the

latter case, the system first passes through the equilibrium position, deviates to the other side of it, and only then approaches the equilibrium position asymptotically. Such motion of a system is called **aperiodic damping** (or an **aperiodic process**).

How a system will return to its equilibrium position (following curve 1 or curve 2) depends on the ratio of the coefficients C_1 and C_2 which, in turn, is determined by the initial conditions [i.e. by the values of the generalized coordinate q_0 and the generalized velocity $v_0 = (\dot{q})_0$ at the initial instant].

Let us establish the conditions in which aperiodic motion has a specific nature. We express the coefficients C_1 and C_2 in terms of q_0 and v_0 . Assuming in (17.4) that $t = 0$, we get

$$q_0 = C_1 + C_2 \quad (17.5)$$

Differentiating (17.4) with respect to time and assuming that $t = 0$ in the expression obtained, we find that

$$v_0 = (\dot{q})_0 = -\alpha_1 C_1 - \alpha_2 C_2 \quad (17.6)$$

It follows from Eqs. (17.5) and (17.6) that

$$C_1 = \frac{\alpha_2 q_0 + v_0}{\alpha_2 - \alpha_1}, \quad C_2 = -\frac{\alpha_1 q_0 + v_0}{\alpha_2 - \alpha_1} \quad (17.7)$$

We equate expression (17.4) to zero:

$$C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t} = 0 \quad (17.8)$$

When aperiodic damping occurs according to curve 2 (see Fig. 17.1), Eq. (17.8) must have a finite positive solution. Solving this equation for t , we obtain

$$t = \frac{1}{\alpha_2 - \alpha_1} \ln \left(-\frac{C_2}{C_1} \right) = \frac{1}{\alpha_2 - \alpha_1} \ln \frac{\alpha_1 q_0 + v_0}{\alpha_2 q_0 + v_0}$$

[we have introduced the values of C_1 and C_2 from (17.7)]. The difference $\alpha_2 - \alpha_1$ is greater than zero (see above). Therefore, t will be positive when the expression inside the logarithm symbol is greater than +1. The latter condition is observed if the expressions $(\alpha_1 q_0 + v_0)$ and $(\alpha_2 q_0 + v_0)$ have the same signs and, in addition, the magnitude of the first expression is greater than that of the second one:

$$\left. \begin{aligned} \operatorname{sgn} (\alpha_1 q_0 + v_0) &= \operatorname{sgn} (\alpha_2 q_0 + v_0) \\ |\alpha_1 q_0 + v_0| &> |\alpha_2 q_0 + v_0| \end{aligned} \right\} \quad (17.9)$$

The coefficients α_1 and α_2 are positive, and $\alpha_2 > \alpha_1$. Hence, for satisfying the second of conditions (17.9), q_0 and v_0 must have different signs. This occurs if the initial velocity is directed towards the equilibrium position [when the system is deflected to the right

($q_0 > 0$), the velocity is directed to the left ($v_0 < 0$), and vice versa. Figure 17.2 shows graphs of the functions $y = \alpha_1 q_0 + v_0$ and $y = \alpha_2 q_0 + v_0$. The graphs have been plotted for $q_0 > 0$, therefore $v_0 < 0$. The values taken on by q_0 are divided into three regions. It is easy to see that both conditions (17.9) are satisfied only in region I, i.e. at q_0 's not exceeding $-v_0/\alpha_2$. In region II, the first of the conditions is not observed, and in region III, the second one.

Hence, aperiodic damping occurs in accordance with curve 2 (see Fig. 17.1) when v_0 and q_0 have different signs and, in addition,

$$|q_0| < \frac{|v_0|}{\alpha_2} \text{ or } |v_0| > \alpha_2 |q_0| \quad (17.10)$$

(we remind our reader that $\alpha_2 = \beta + \sqrt{\beta^2 - \omega_0^2}$).

Special attention must be given to the case when characteristic equation (17.2) has multiple roots. This occurs provided that $\beta^2 = \omega_0^2$. Consequently, $\lambda_1 = \lambda_2 = -\beta$. According to formula (V.11), in this case the general solution of Eq. (17.1) is

$$q = C_1 e^{-\beta t} + C_2 t e^{-\beta t} = (C_1 + C_2 t) e^{-\beta t}$$

After the relevant calculations, we find that

$$C_1 = q_0 \text{ and } C_2 = \beta q_0 + v_0$$

From the condition $q = 0$, we get (except for $t = \infty$) the value

$$t = -\frac{C_1}{C_2} = -\frac{q_0}{\beta q_0 + v_0}$$

It will be positive if

$$\frac{q_0}{\beta q_0 + v_0} < 0 \text{ or } \frac{\beta q_0}{\beta q_0 + v_0} < 0$$

(multiplication by β does not change the sign of a quantity because $\beta > 0$). The last condition is observed when the sign of v_0 is opposite to that of q_0 and, in addition,

$$|v_0| > \beta |q_0| \quad (17.11)$$

Hence, with multiple roots, aperiodic damping can also occur either monotonously (see the curve 1 in Fig. 17.1) or with passing through the equilibrium position (see the curve 2 in Fig. 17.1). The latter case occurs if a system brought out of its equilibrium position

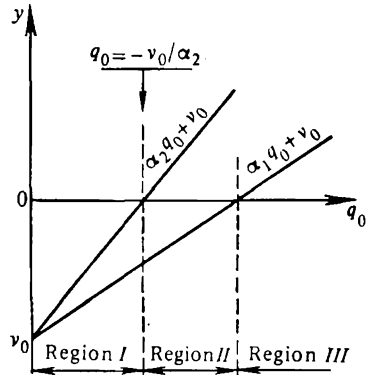


Fig. 17.2.

by q_0 receives an impetus towards the equilibrium position, imparting a sufficiently high initial velocity to it [a velocity satisfying condition (17.11); when the roots are different, the velocity must comply with condition (17.10)].

18. Forced Oscillations

Assume that the system treated in the preceding section experiences the periodically changing external generalized force

$$Q^* = Q_0 \cos(\omega t + \alpha) \quad (18.1)$$

which we shall call the driving force for brevity's sake. Lagrange's equation (4.15) therefore becomes

$$\mu \ddot{q} + \kappa \dot{q} = -r\dot{q} + Q_0 \cos(\omega t + \alpha)$$

Let us transform it to the form

$$\ddot{q} + 2\beta \dot{q} + \omega_0^2 q = f_0 \cos(\omega t + \alpha) \quad (18.2)$$

where $f_0 = Q_0/\mu$; the other quantities are explained in the preceding sections.

We have arrived at a linear non-homogeneous differential equation with constant coefficients. According to theorem (V.6), we can obtain its general solution by adding a particular solution of Eq. (18.2) to the general solution of the corresponding homogeneous equation, i.e. to the function (17.3). To find the particular solution, let us proceed in accordance with what is said at the end of Appendix V, namely, let us add the imaginary function $if_0 \sin(\omega t + \alpha)$ to the right-hand side of (18.2) and seek the complex solution \hat{q} of the equation obtained; after finding \hat{q} , we shall take its real part, and the latter will be the solution of Eq. (18.2). Hence, we shall solve the equation

$$\ddot{\hat{q}} + 2\beta \dot{\hat{q}} + \omega_0^2 \hat{q} = f_0 [\cos(\omega t + \alpha) + i \sin(\omega t + \alpha)]$$

Its right-hand side can be written as

$$f_0 e^{i(\omega t + \alpha)} = f_0 e^{i\omega t}$$

where

$$\hat{f}_0 = f_0 e^{i\alpha} \quad (18.3)$$

is the complex amplitude¹ of the driving force (more exactly, the force divided by μ , but for brevity's sake we shall call Q^*/μ simply a force). The differential equation written in the new notation will be

$$\ddot{\hat{q}} + 2\beta \dot{\hat{q}} + \omega_0^2 \hat{q} = \hat{f}_0 e^{i\omega t} \quad (18.4)$$

(we have omitted the cap over q to avoid complicated symbols).

¹ Compare with (16.10).

We shall seek the solution of Eq. (18.4) in the form

$$q = \hat{a}e^{i\omega t} \quad (18.5)$$

where \hat{a} is the complex amplitude of the oscillation. Differentiation with respect to t yields

$$\dot{q} = i\omega\hat{a}e^{i\omega t}, \quad \ddot{q} = (i\omega)^2\hat{a}e^{i\omega t} = -\omega^2\hat{a}e^{i\omega t} \quad (18.6)$$

We see that in the complex representation of harmonically varying quantities time differentiation consists in multiplying the quantities by $i\omega$ (in integration—in dividing them by $i\omega$).

Introducing expressions (18.5) and (18.6) into Eq. (18.4) and cancelling the common factor $e^{i\omega t}$, we obtain the equation

$$-\omega^2\hat{a} + 2i\beta\omega\hat{a} + \omega_0^2\hat{a} = \hat{f}_0$$

from which we find

$$\hat{a} = \frac{\hat{f}_0}{(\omega_0^2 - \omega^2) + 2i\beta\omega}$$

We represent the complex number in the denominator as

$$(\omega_0^2 - \omega^2) + 2i\beta\omega = \rho e^{i\varphi} \quad (18.7)$$

where ρ is the modulus and φ is the argument of this number. Therefore,

$$\hat{a} = \frac{\hat{f}_0}{\rho e^{i\varphi}} = \frac{1}{\rho} e^{-i\varphi} \hat{f}_0 \quad (18.8)$$

It follows from (18.7)¹ that

$$\rho = \sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}, \quad \tan \varphi = \frac{2\beta\omega}{\omega_0^2 - \omega^2} \quad (18.9)$$

Using the values of ρ and \hat{f}_0 [see (18.3)] in (18.8), we get the following expression for the complex amplitude:

$$\hat{a} = \frac{\hat{f}_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} e^{i(\alpha - \varphi)} = a e^{i(\alpha - \varphi)}$$

Finally, introducing the value of \hat{a} into formula (18.5), we find the complex expression for \hat{q} :

$$\hat{q} = a e^{i(\omega t + \alpha - \varphi)}$$

¹ Recall that a complex number can be depicted by a point P on a plane. The abscissa x of this point equals the real part of the number, and the ordinate y equals its imaginary part. The modulus of a complex number equals the modulus ρ of the position vector of the point P , while the argument φ is the angle made by the position vector and the axis of abscissas. It thus follows that $\rho = \sqrt{x^2 + y^2}$ and $\tan \varphi = y/x$.

Its real part coincides with the expression for steady-state forced oscillations known from the general course of physics:

$$q = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \cos(\omega t + \alpha - \varphi) \quad (18.10)$$

(in textbooks of general physics, it is usually assumed that $\alpha = 0$).

We obtain the general solution of Eq. (18.2) by summation of the functions (17.3) and (18.10). We shall not stop off to analyse this solution and consider the phenomenon of resonance because this is done in sufficient detail in general courses of physics.

19. Oscillations of a System with Many Degrees of Freedom ¹

Consider a conservative system with s degrees of freedom and having a position of stable equilibrium. In this position, the potential energy of the system $U = U(q_1, q_2, \dots, q_s)$ has a minimum. We shall measure the generalized coordinates q_i from the equilibrium position. Bearing in mind that we shall limit ourselves to small-amplitude oscillations, let us expand the potential energy in powers of q_i , disregarding the terms of the higher orders of smallness:

$$U = U_0 + \sum_i \left(\frac{\partial U}{\partial q_i} \right)_0 q_i + \frac{1}{2} \sum_{i, k} \left(\frac{\partial^2 U}{\partial q_i \partial q_k} \right)_0 q_i q_k$$

In the equilibrium position, all the generalized forces $Q_i = -(\partial U / \partial q_i)_0$ vanish. We also assume the energy U_0 to vanish. The expression for the potential energy can thus be written as

$$U = \frac{1}{2} \sum_{i, k} \kappa_{ik} q_i q_k \quad (19.1)$$

where

$$\kappa_{ik} = \kappa_{ki} = \left(\frac{\partial^2 U}{\partial q_i \partial q_k} \right)_0$$

are positive quantities (at a minimum, the second derivatives are positive). Since U is measured from its minimum value taken as zero, quadratic form (19.1) is positive definite.

With stationary constraints, the kinetic energy is determined by a positive definite quadratic form of the variables \dot{q}_i [see (5.10)]:

$$T = \frac{1}{2} \sum_{i, k} \mu_{ik} \dot{q}_i \dot{q}_k \quad (19.2)$$

where

$$\mu_{ik} = \gamma_{ik} (0)$$

¹ Before beginning to read this section, acquaint yourself with Appendices VII, VIII, and IX.

are the zero terms of the expansion of the coefficients $\gamma_{ik}(q)$. By formula (5.8), $\gamma_{ik} = \gamma_{ki}$, therefore $\mu_{ik} = \mu_{ki}$.

Subtraction of expression (19.1) from (19.2) yields the Lagrangian:

$$L = \frac{1}{2} \sum_{i, k} \mu_{ik} \dot{q}_i \dot{q}_k - \frac{1}{2} \sum_{i, k} \kappa_{ik} q_i q_k \quad (19.3)$$

To find the derivatives of L with respect to q_i and \dot{q}_i , we write the expression of the total differential of the function (19.3):

$$dL = \frac{1}{2} \sum_{i, k} \mu_{ik} \dot{q}_i d\dot{q}_k + \frac{1}{2} \sum_{i, k} \mu_{ik} \dot{q}_k d\dot{q}_i - \frac{1}{2} \sum_{i, k} \kappa_{ik} q_i dq_k - \frac{1}{2} \sum_{i, k} \kappa_{ik} q_k dq_i$$

The subscripts i and k are dummy ones, therefore any letter may be used for either of them. Taking advantage of this, let us exchange the places of the subscripts i and k in the first and third sums:

$$\begin{aligned} dL &= \frac{1}{2} \sum_{i, k} \mu_{ki} \dot{q}_k d\dot{q}_i + \frac{1}{2} \sum_{i, k} \mu_{ik} \dot{q}_k d\dot{q}_i \\ &\quad - \frac{1}{2} \sum_{i, k} \kappa_{ki} q_k dq_i - \frac{1}{2} \sum_{i, k} \kappa_{ik} q_k dq_i = \sum_{i, k} \mu_{ik} \dot{q}_k d\dot{q}_i - \sum_{i, k} \kappa_{ik} q_k dq_i \end{aligned}$$

(recall that $\mu_{ki} = \mu_{ik}$, and $\kappa_{ki} = \kappa_{ik}$). The expression we have obtained can be written as

$$dL = \sum_i d\dot{q}_i \left(\sum_k \mu_{ik} \dot{q}_k \right) - \sum_i dq_i \left(\sum_k \kappa_{ik} q_k \right) \quad (19.4)$$

In an expression for the total differential of a function of several variables, the factor of a differential of a variable equals the partial derivative of the function with respect to this variable. It thus follows from (19.4) that

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_k \mu_{ik} \dot{q}_k, \quad \frac{\partial L}{\partial q_i} = - \sum_k \kappa_{ik} q_k$$

Since the quantities μ_{ik} are constants, the derivative $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \sum_k \mu_{ik} \ddot{q}_k$. Hence, Lagrange's equations have the form

$$\sum_k \mu_{ik} \ddot{q}_k + \sum_k \kappa_{ik} q_k = 0 \quad (i = 1, 2, \dots, s) \quad (19.5)$$

[compare with Eq. (16.4) for one variable].

We have arrived at a system of linear homogeneous differential equations with constant coefficients. Let us attempt to seek the unknown functions $q_k(t)$ in the form [compare with (16.5)]

$$q_k = C_k e^{i\omega t} \quad (19.6)$$

where C_k are complex constants that have to be determined. The functions (19.6) are complex, whereas the generalized coordinates are real. Consequently, upon completing our calculations, we shall have to take the real parts of functions (19.6) (see Appendix V).

The introduction of expressions (19.6) into Eq. (19.5) yields

$$\sum_k \mu_{ik} (-\omega^2) C_k e^{i\omega t} + \sum_k \kappa_{ik} C_k e^{i\omega t} = 0 \quad (i = 1, 2, \dots, s)$$

Cancelling $e^{i\omega t}$ in all the equations, we obtain

$$\sum_k (\kappa_{ik} - \omega^2 \mu_{ik}) C_k = 0 \quad (19.7)$$

We have arrived at a system of s linear homogeneous algebraic equations with the unknowns C_1, C_2, \dots, C_s . For this system to have a non-zero solution, it is necessary and sufficient that its determinant be zero:

$$\begin{vmatrix} \kappa_{11} - \omega^2 \mu_{11} & \kappa_{12} - \omega^2 \mu_{12} & \dots & \kappa_{1s} - \omega^2 \mu_{1s} \\ \kappa_{21} - \omega^2 \mu_{21} & \kappa_{22} - \omega^2 \mu_{22} & \dots & \kappa_{2s} - \omega^2 \mu_{2s} \\ \dots & \dots & \dots & \dots \\ \kappa_{s1} - \omega^2 \mu_{s1} & \kappa_{s2} - \omega^2 \mu_{s2} & \dots & \kappa_{ss} - \omega^2 \mu_{ss} \end{vmatrix} = 0 \quad (19.8)$$

[see the text following formula (VIII.26) in Appendix VIII].

Equation (19.8) is known as a **characteristic equation**. It is an equation of degree s relative to ω^2 . In the general case, this equation has s different¹ real positive roots: $\omega_1^2, \omega_2^2, \dots, \omega_s^2$. The quantities ω_α ($\alpha = 1, 2, \dots, s$) found in this way are called the **natural frequencies** of the system.

Let us prove that the roots of Eq. (19.8) are real and positive. For this purpose, we multiply each of Eqs. (19.7) by C_i^* (i.e. by a quantity that is the complex conjugate of the coefficient C_i) and then summate all the equations. The result is

$$\sum_{i, k} (\kappa_{ik} - \omega^2 \mu_{ik}) C_i^* C_k = 0$$

or

$$\sum_{i, k} \kappa_{ik} C_i^* C_k - \omega^2 \sum_{i, k} \mu_{ik} C_i^* C_k = 0$$

whence

$$\omega^2 = \frac{\sum_{i, k} \kappa_{ik} C_i^* C_k}{\sum_{i, k} \mu_{ik} C_i^* C_k} \quad (19.9)$$

The numerator and denominator of Eq. (19.9) contain quadratic forms like (IX.24). It is shown in Appendix IX that such a form

¹ Multiple roots may be obtained in particular cases.

equals the sum of the quadratic forms, $\sum_{i,k} \kappa_{ik} a_i a_k + \sum_{i,k} \kappa_{ik} b_i b_k$, and correspondingly $\sum_{i,k} \mu_{ik} a_i a_k + \sum_{i,k} \mu_{ik} b_i b_k$ (a_i is the real part, and b_i is the imaginary part of C_i). The latter forms, in turn, are, first, evidently real and, second, positive definite [see (19.1) and (19.2)]. We have thus proved that the numerator and denominator of Eq. (19.9) and, consequently, ω^2 are real and positive.

Thus, having solved the characteristic equation (19.8), we find s natural frequencies of the system: $\omega_1, \omega_2, \dots, \omega_s$. Introducing in turn the values ω_α^2 into the system of equations (19.7) and solving the system, we find C_k 's corresponding to different ω_α 's. If the matrix of the system (19.7) has the rank $s - 1$ (which is usually the case), by (VIII.28) the solutions of the system are

$$C_k^{(\alpha)} = c_\alpha A_{mk}^{(\alpha)}$$

where c_α is an arbitrary complex constant, and $A_{mk}^{(\alpha)}$ is the signed minor of the element $\kappa_{mk} - \omega_\alpha^2 \mu_{mk}$ in the determinant of the system (m is chosen arbitrarily but with at least one $A_{mk}^{(\alpha)}$ being non-zero). Since all the elements of this determinant are real, the quantities $A_{mk}^{(\alpha)}$ are also real.

Hence, for each generalized coordinate q_k , we obtain s different solutions of the form

$$q_k = c_\alpha A_{mk}^{(\alpha)} e^{i\omega_\alpha t} \quad (\alpha = 1, 2, \dots, s) \quad (19.10)$$

where $A_{mk}^{(\alpha)}$ are real constants determined by the values of the coefficients κ_{ik} and μ_{ik} , and also of the frequencies ω_α .

We obtain the general solution by summation of all expressions (19.10):

$$q_k = \sum_{\alpha} c_\alpha A_{mk}^{(\alpha)} e^{i\omega_\alpha t}$$

Passing over to the real part of this expression, we obtain

$$q_k = \operatorname{Re} \left\{ \sum_{\alpha} c_\alpha A_{mk}^{(\alpha)} e^{i\omega_\alpha t} \right\} = \sum_{\alpha} A_{mk}^{(\alpha)} \operatorname{Re} \{ c_\alpha e^{i\omega_\alpha t} \}$$

Finally, representing c_α as $a_\alpha e^{i\delta_\alpha}$ (here a_α is the modulus of c_α , i.e. a real positive quantity), we arrive at the expression

$$q_k = \sum_{\alpha=1}^s A_{mk}^{(\alpha)} a_\alpha \cos(\omega_\alpha t + \delta_\alpha) \quad (19.11)$$

Consequently, the change in each generalized coordinate q_k with time is the superposition of s harmonic oscillations whose frequencies equal the natural frequencies of the system. The quantities a_α and δ_α are determined from the initial conditions.

Expressions (19.11) can be greatly simplified with a special selection of the generalized coordinates. It is shown in Appendix IX that when we have two quadratic forms—one from the variables \dot{q}_k and the other from the variables q_k , the first of them being positive definite, a linear transformation of the variables q_k exists such that reduces both forms to a diagonal type [see diagram (IX.37)]. Let us pass over with the aid of such a transformation from the variables q_k to the variables ξ_k . The quadratic forms (19.1) and (19.2) will now become diagonal:

$$T = \frac{1}{2} \sum_k \dot{\xi}_k^2, \quad U = \frac{1}{2} \sum_k \lambda_k \xi_k^2 \quad (19.12)$$

The Lagrangian will be as follows in the new variables:

$$L = \frac{1}{2} \sum_k \dot{\xi}_k^2 - \frac{1}{2} \sum_k \lambda_k \xi_k^2$$

and Lagrange's equations will be

$$\ddot{\xi}_k + \lambda_k \xi_k = 0 \quad (k = 1, 2, \dots, s) \quad (19.13)$$

The equations of motion in the coordinates ξ_k thus split up into s independent equations each of which is identical to Eq. (16.4). We must note that owing to the positive definiteness of the quadratic form for the potential energy U , all the coefficients λ_k are positive. They can therefore be written as

$$\lambda_k = \omega_k^2$$

where ω_k are real quantities.

Let us write the solutions of Eqs. (19.13):

$$\xi_k = a_k \cos(\omega_k t + \delta_k) \quad (k = 1, 2, \dots, s) \quad (19.14)$$

[see formula (16.7)].

We have found that the generalized coordinates ξ_k perform a simple harmonic oscillation independently of one another, and each with its own frequency ω_k . The generalized coordinates determined in this way are called **normal** (or **principal**), and the simple harmonic oscillations they perform—**normal oscillations** of the system.

We must note that the normal coordinates ξ_k are related to the arbitrary generalized coordinates q_k by means of linear transformations, i.e. transformations of the form

$$\xi_k = \sum_i b_{ki} q_i \quad (19.15)$$

Hence, ξ_k can be obtained as a linear combination of the coordinates q_i .

20. Coupled Pendulums

Consider the small-amplitude oscillations of a system consisting of two identical simple pendulums connected by a weightless spring (Fig. 20.1). Assume that the pendulums can oscillate only in the plane of the drawing so that the system has two degrees of freedom. We choose φ_1 and φ_2 —the angles of deflection of the pendulums from the vertical direction—as the generalized coordinates. The length of each pendulum is l and its mass is m . The ends of the spring are fastened on the pendulum rods at a distance b from the point of suspension. The spring is chosen so that when $\varphi_1 = \varphi_2$, its tension is zero.

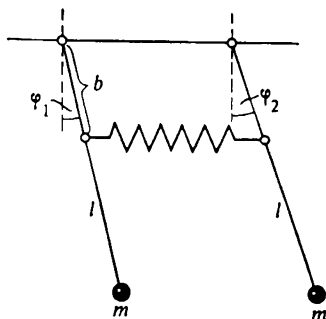


Fig. 20.1.

Let us write an expression for the potential energy U of the system, assuming that U is zero in the equilibrium position:

$$U = mgl(1 - \cos \varphi_1) + mgl(1 - \cos \varphi_2) + \frac{1}{2} k (b \sin \varphi_2 - b \sin \varphi_1)^2$$

For small-amplitude oscillations, we can assume that $\sin \varphi = \varphi$ and $\cos \varphi = \sqrt{1 - \sin^2 \varphi} = \sqrt{1 - \varphi^2} = 1 - \frac{1}{2} \varphi^2$. The expression for U thus becomes

$$U = \frac{1}{2} mgl\varphi_1^2 + \frac{1}{2} mgl\varphi_2^2 + \frac{1}{2} kb^2 (\varphi_2 - \varphi_1)^2$$

$$= \frac{1}{2} [(mgl + kb^2) \varphi_1^2 - kb^2 \varphi_1 \varphi_2 - kb^2 \varphi_2 \varphi_1 + (mgl + kb^2) \varphi_2^2] \quad (20.1)$$

The kinetic energy in the same approximation is

$$T = \frac{1}{2} [ml^2 \dot{\varphi}_1^2 + ml^2 \dot{\varphi}_2^2] \quad (20.2)$$

A comparison of expressions (20.1) and (20.2) with expressions (19.1) and (19.2) yields the following values for the coefficients κ_{ik} and μ_{ik} :

$$\begin{aligned} \kappa_{11} = \kappa_{22} &= mgl + kb^2, & \kappa_{12} = \kappa_{21} &= -kb^2 \\ \mu_{11} = \mu_{22} &= ml^2, & \mu_{12} = \mu_{21} &= 0 \end{aligned} \quad (20.3)$$

The introduction of these values of the coefficients into Eq. (19.5) leads to the differential equations

$$\left. \begin{aligned} ml^2 \ddot{\varphi}_1 + (mgl + kb^2) \varphi_1 - kb^2 \varphi_2 &= 0 \\ ml^2 \ddot{\varphi}_2 - kb^2 \varphi_1 + (mgl + kb^2) \varphi_2 &= 0 \end{aligned} \right\} \quad (20.4)$$

We shall seek the solutions of these equations in the form

$$\varphi_1 = C_1 e^{i\omega t} \quad \text{and} \quad \varphi_2 = C_2 e^{i\omega t} \quad (20.5)$$

Let us substitute these expressions into Eq. (20.4). After cancelling $e^{i\omega t}$ and combining similar terms, we get a system of equations for determining the constants C_1 and C_2 :

$$\left. \begin{aligned} (mgl + kb^2 - ml^2\omega^2) C_1 - kb^2 C_2 &= 0 \\ -kb^2 C_1 + (mgl + kb^2 - ml^2\omega^2) C_2 &= 0 \end{aligned} \right\} \quad (20.6)$$

For this system to have a non-zero solution, its determinant must equal zero:

$$\begin{vmatrix} (mgl + kb^2 - ml^2\omega^2) & (-kb^2) \\ (-kb^2) & (mgl + kb^2 - ml^2\omega^2) \end{vmatrix} = 0$$

i.e. the following condition must be satisfied:

$$(mgl + kb^2 - ml^2\omega^2)^2 - (-kb^2)^2 = 0.$$

The latter equation can be written as follows after simple transformations:

$$(\omega^2)^2 - 2 \left(\frac{g}{l} + \frac{k}{m} \frac{b^2}{l^2} \right) \omega^2 + \left[\frac{g^2}{l^2} + 2 \left(\frac{g}{l} \frac{k}{m} \frac{b^2}{l^2} \right) \right] = 0$$

We have arrived at a quadratic equation relative to ω^2 . The roots of this equation are

$$\omega_1^2 = \frac{g}{l} \quad \text{and} \quad \omega_2^2 = \frac{g}{l} + 2 \frac{k}{m} \frac{b^2}{l^2}.$$

Consequently, the natural frequencies of the system will be

$$\omega_1 = \sqrt{\frac{g}{l}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{g}{l} + 2 \frac{k}{m} \frac{b^2}{l^2}} \quad (20.7)$$

Let us introduce the square of the first natural frequency, i.e. ω_1^2 into Eq. (20.6) instead of ω^2 . After simplification, system (20.6) becomes

$$\begin{aligned} kb^2 C_1 - kb^2 C_2 &= 0 \\ -kb^2 C_1 + kb^2 C_2 &= 0 \end{aligned}$$

The solutions of this system are obvious:

$$C_1 = C_2 = c_1 = a_1 e^{i\delta_1} \quad (20.8)$$

where c_1 is an arbitrary complex constant, a_1 is its modulus, and δ_1 is its argument.

The introduction of (20.8) into (20.5) yields complex solutions of differential equations (20.4) corresponding to the frequency ω_1 :

$$\varphi_1^{(1)} = c_1 e^{i\omega_1 t} = a_1 e^{i(\omega_1 t + \delta_1)}$$

$$\varphi_2^{(1)} = c_1 e^{i\omega_1 t} = a_1 e^{i(\omega_1 t + \delta_1)}$$

Taking the real part of the functions we have found, we obtain

$$\varphi_1^{(1)} = a_1 \cos(\omega_1 t + \delta_1), \quad \varphi_2^{(1)} = a_1 \cos(\omega_1 t + \delta_1) \quad (20.9)$$

Now let us introduce the square of the second natural frequency, i.e. ω_2^2 , into Eq. (20.6) instead of ω^2 . The result is

$$\begin{aligned} -kb^2 C_1 - kb^2 C_2 &= 0 \\ -kb^2 C_1 - kb^2 C_2 &= 0 \end{aligned}$$

The system is satisfied by the values

$$C_1 = -C_2 = c_2 = a_2 e^{i\delta_2}$$

The functions (20.5) will accordingly be

$$\left. \begin{aligned} \varphi_1^{(2)} &= c_2 e^{i\omega_2 t} = a_2 e^{i(\omega_2 t + \delta_2)} \\ \varphi_2^{(2)} &= -c_2 e^{i\omega_2 t} = -a_2 e^{i(\omega_2 t + \delta_2)} \end{aligned} \right\}$$

and their real parts will be

$$\left. \begin{aligned} \varphi_1^{(2)} &= a_2 \cos(\omega_2 t + \delta_2) \\ \varphi_2^{(2)} &= -a_2 \cos(\omega_2 t + \delta_2) \end{aligned} \right\} \quad (20.10)$$

We obtain a general solution of system (20.4) by the summation of solutions (20.9) and (20.10). Consequently,

$$\left. \begin{aligned} \varphi_1 &= \varphi_1^{(1)} + \varphi_1^{(2)} = a_1 \cos(\omega_1 t + \delta_1) + a_2 \cos(\omega_2 t + \delta_2) \\ \varphi_2 &= \varphi_2^{(1)} + \varphi_2^{(2)} = a_1 \cos(\omega_1 t + \delta_1) - a_2 \cos(\omega_2 t + \delta_2) \end{aligned} \right\} \quad (20.11)$$

Let us go over from the generalized coordinates φ_1 and φ_2 to the new variables ξ_1 and ξ_2 , which we shall determine as follows:

$$\xi_1 = \frac{1}{2}(\varphi_1 + \varphi_2) \quad \text{and} \quad \xi_2 = \frac{1}{2}(\varphi_1 - \varphi_2)$$

With a view to (20.11), we obtain

$$\left. \begin{aligned} \xi_1 &= a_1 \cos(\omega_1 t + \delta_1) \\ \xi_2 &= a_2 \cos(\omega_2 t + \delta_2) \end{aligned} \right\} \quad (20.12)$$

The variables ξ_1 and ξ_2 are thus normal coordinates of the system of coupled pendulums being considered. The generalized coordinates φ_1 and φ_2 are expressed in terms of ξ_1 and ξ_2 with the aid of the linear equations

$$\varphi_1 = \xi_1 + \xi_2 \quad \text{and} \quad \varphi_2 = \xi_1 - \xi_2 \quad (20.13)$$

Assume that only the first normal oscillation is performed in the system. This signifies that $\xi_2 \equiv 0$. Inspection of (20.13) shows that in this case

$$\varphi_1 = \varphi_2 = \xi_1 = a_1 \cos(\omega_1 t + \delta_1)$$

i.e. both pendulums oscillate like a single whole with the frequency ω_1 , being at each instant deflected to the same side through the same angle (Fig. 20.2a). The spring is not deformed so that each pendulum oscillates as if the other one were absent ($\omega_1 = \sqrt{g/l}$).

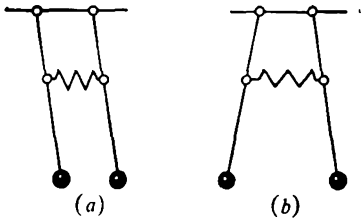


Fig. 20.2.

Now assume that only the second normal oscillation is being performed in the system. A glance at (20.13) shows that in this case

$$\varphi_1 = -\varphi_2 = \xi_2 = a_2 \cos(\omega_2 t + \delta_2)$$

At each instant, the pendulums are deflected through an identical angle, but in opposite directions (Fig. 20.2b).

The connection between the pendulums can be characterized with the aid of the spring constant k . Let us call the latter the coupling coefficient. Consider the case of a weak connection, i.e. a small k . If $(k/m) \ll (g/l)$, the difference between the natural frequencies will be much smaller than the frequencies themselves:

$$\omega_2 - \omega_1 \ll \omega_1 \quad (20.14)$$

Let us deflect the first pendulum through the angle $\varphi_{10} = \alpha$, keeping the second one at its zero position. Now let the system oscillate. The initial conditions in this case will obviously be

$$\varphi_{10} = \alpha, \quad \varphi_{20} = 0, \quad (\dot{\varphi}_1)_0 = 0, \quad (\dot{\varphi}_2)_0 = 0$$

We shall find the values of the constants a_1 , a_2 , δ_1 , and δ_2 . For this purpose, we assume that $t = 0$ in (20.11). The result is

$$\begin{cases} \varphi_{10} = \alpha = a_1 \cos \delta_1 + a_2 \cos \delta_2 \\ \varphi_{20} = 0 = a_1 \cos \delta_1 - a_2 \cos \delta_2 \end{cases} \quad (20.15)$$

Now let us differentiate expressions (20.11) with respect to time and assume that $t = 0$ in the formulas obtained. This leads to the expressions

$$\begin{cases} (\dot{\varphi}_1)_0 = 0 = -a_1 \omega_1 \sin \delta_1 - a_2 \omega_2 \sin \delta_2 \\ (\dot{\varphi}_2)_0 = 0 = -a_1 \omega_1 \sin \delta_1 + a_2 \omega_2 \sin \delta_2 \end{cases} \quad (20.16)$$

Solving Eqs. (20.15) and (20.16) simultaneously, we find that

$$a_1 = a_2 = \frac{\alpha}{2} \quad \text{and} \quad \delta_1 = \delta_2 = 0$$

Hence, in the case being considered, the oscillations have the form

$$\varphi_1 = \frac{\alpha}{2} (\cos \omega_1 t + \cos \omega_2 t) = \alpha \cos \frac{\omega_2 - \omega_1}{2} t \cdot \cos \frac{\omega_2 + \omega_1}{2} t$$

$$\varphi_2 = \frac{\alpha}{2} (\cos \omega_1 t - \cos \omega_2 t) = \alpha \sin \frac{\omega_2 - \omega_1}{2} t \cdot \sin \frac{\omega_2 + \omega_1}{2} t$$

With a weak coupling, we have $(\omega_2 - \omega_1) \ll (\omega_2 + \omega_1)$ [see (20.14)]. We may thus consider that each of the pendulums performs harmonic oscillation at the frequency $(\omega_2 + \omega_1)/2 \approx \omega_1$ with a slowly varying amplitude. Hence, the motion of each of the pendulums has the nature of beats. The amplitudes change with a phase shift of $\pi/2$. When the amplitude of one of the pendulums reaches its maximum value, the amplitude of the second one vanishes, and vice versa. In the process of oscillations, energy is pumped, as it were, from one pendulum to the other.

When only one normal oscillation ξ_1 or ξ_2 is produced, no transition of energy from one pendulum to the other occurs.

Chapter V

MECHANICS OF A RIGID BODY

21. Kinematics of a Rigid Body

By dividing a continuous rigid body into elementary volumes of mass ρdV (here ρ is the density of the body), it can be represented as a system of particles with rigid constraints.

A rigid body is known to have six degrees of freedom—three translational and three rotational ones. To describe the motion of a rigid body, let us choose the inertial reference frame K (with the axes X_1, X_2, X_3) which we shall consider to be stationary. We shall rigidly connect to the body another frame K' (with the axes x_1, x_2, x_3) and place its origin at the point A of the body. It is convenient to take the three coordinates of the origin of the frame K' (the position vector \mathbf{R}_A corresponds to them) and the three angles characterizing the orientation of the axes x_1, x_2, x_3 relative to the axes X_1, X_2, X_3 as the generalized coordinates determining the position of the body. These axes make nine angles with one another, but only three of them are independent; the other six can be expressed through the values of the first three¹. It is customary practice to use the Euler angles φ, θ, ψ (see Sec. 22) as the three angles determining the mutual orientation of the axes of the frames K and K' .

Any elementary displacement of a rigid body can be represented as the sum of a translational displacement when all its points are displaced over the same distance $d\mathbf{R}_A$ and rotation through the angle $d\Phi$ about an axis passing through the point A .

Since the velocities \mathbf{v} of the points of the body in the frame K' are zero, formula (15.3) for the velocity of a point whose position in the frame K' is determined by the position vector $\mathbf{r}_{(A)}$ ² acquires

¹ There are six relations between the cosines of these angles [see formula (VI.39)]:

$$\sum_m \alpha_{im} \alpha_{km} = \delta_{ik} \quad (i, k = 1, 2, 3; i \leq k)$$

² $[\alpha_{ik} = \cos(x_i, X_k)]$.

Let us agree on notation. In this chapter, we shall use indices of two kinds: (1) without parentheses, and (2) in parentheses. Indices without parentheses will indicate a particle or point which the given quantity relates to. For instance, m_α is the mass of a particle whose number is α , \mathbf{r}_α is the position vector of the same particle, and \mathbf{R}_A is the position vector of the point A .

Indices in parentheses will indicate the point from which a position vector emerges, or the point relative to which a moment or angular momentum is calculated.

the form

$$\mathbf{V} = \mathbf{V}_A + [\boldsymbol{\omega}_{(A)}, \mathbf{r}_{(A)}] \quad (21.1)$$

where \mathbf{V}_A is the translational velocity of the body (the velocity of the point A observed in the frame K), and $\boldsymbol{\omega}_{(A)} = d\Phi/dt$ is the angular velocity of rotation of the body about an axis passing through the point A . The first term in this formula is the same for all the points of a body, the second is a position function.

If we had placed the origin of the frame K' at another point of the body, say at a point B , formula (21.1) would be as follows:

$$\mathbf{V} = \mathbf{V}_B + [\boldsymbol{\omega}_{(B)}, \mathbf{r}_{(B)}] \quad (21.2)$$

where \mathbf{V}_B is the velocity of the point B observed in the frame K , and $\boldsymbol{\omega}_{(B)}$ is the angular velocity of rotation of the body about an axis passing through the point B .

The position of an arbitrary point of the body in the frame K is determined by the same position vector in both cases:

$$\mathbf{R} = \mathbf{R}_A + \mathbf{r}_{(A)} = \mathbf{R}_B + \mathbf{r}_{(B)}$$

It thus follows that the position vector $\mathbf{r}_{(B)}$ can be represented as

$$\mathbf{r}_{(B)} = \mathbf{a} + \mathbf{r}_{(A)} \quad (21.3)$$

where $\mathbf{a} = \mathbf{R}_A - \mathbf{R}_B$ is a position vector from the point B to the point A , i.e. a quantity not depending on which point of the body we write formula (21.3) for.

Using the value given by (21.3) in formula (21.2), we obtain

$$\mathbf{V} = \mathbf{V}_B + [\boldsymbol{\omega}_{(B)}, \mathbf{a}] + [\boldsymbol{\omega}_{(B)}, \mathbf{r}_{(A)}] \quad (21.4)$$

The first two terms on the right-hand side of (21.4) are identical for all the points of the body, while the third term is a position function.

Formulas (21.1) and (21.4) determine the same quantity—the velocity of the point of the body being considered in the frame K .

culated, etc. Depending on the circumstances, we shall use these indices either as subscripts or superscripts on the relevant symbol. For example, $\mathbf{r}_{(A)}$ or $\mathbf{r}^{(A)}$ will stand for a position vector emerging from the point A ; $\mathbf{M}_{(A)}$ or $\mathbf{M}^{(A)}$, for the angular momentum relative to the point A . The absence of an index in parentheses at the symbol \mathbf{r} or \mathbf{M} will signify that the relevant quantity is taken relative to the centre of mass C of the body. Hence, we shall designate a position vector emerging from the point C either by the symbol $\mathbf{r}_{(C)}$ or simply by \mathbf{r} , the angular momentum relative to the point C , by the symbols $\mathbf{M}_{(C)}$ and \mathbf{M} , and so on.

We shall use the symbol \mathbf{R} only in one quite definite case—to denote a position vector emerging from the origin of the stationary reference frame K (with the axes X, Y, Z). Therefore, there is no need to use an index in parentheses in the given case, and we shall not write it.

We shall use lower case letters (\mathbf{r}, x, y, z , etc.) to denote position vectors drawn from the origin of the reference frame K' rigidly connected to a body, the coordinates in the frame K' , etc.

Consequently, at any $\mathbf{r}_{(A)}$, the right-hand sides of these formulas must coincide. This is possible provided that

$$\mathbf{V}_A = \mathbf{V}_B + [\boldsymbol{\omega}_{(B)}, \mathbf{a}] \quad (21.5)$$

$$[\boldsymbol{\omega}_{(A)}, \mathbf{r}_{(A)}] \equiv [\boldsymbol{\omega}_{(B)}, \mathbf{r}_{(A)}] \quad (21.6)$$

[the identity sign stresses that equality must hold at any values of $\mathbf{r}_{(A)}$].

A glance at identity (21.6) shows that

$$\boldsymbol{\omega}_{(A)} = \boldsymbol{\omega}_{(B)}$$

i.e. that the angular velocity of rotation about any axis is the same, and we can speak simply of the angular velocity $\boldsymbol{\omega}$ of the body regardless of our choice of the reference frame K' . The translational velocity, as can be seen from relation (21.5), does not have an absolute nature, however; it depends on the position of the origin of the frame K' (that is, $\mathbf{V}_A \neq \mathbf{V}_B$).

Suppressing the superfluous subscript on $\boldsymbol{\omega}$, let us write relation (21.5) as follows:

$$\mathbf{V}_B = \mathbf{V}_A - [\boldsymbol{\omega}\mathbf{a}] \quad (21.7)$$

Two cases are possible: (1) the vectors \mathbf{V}_A and $\boldsymbol{\omega}$ are mutually perpendicular, and (2) the vectors \mathbf{V}_A and $\boldsymbol{\omega}$ make an angle differing from $\pi/2$. It is easy to see that in the first case the vectors \mathbf{V}_A and $[\boldsymbol{\omega}\mathbf{a}]$ are coplanar. Consequently, the vectors \mathbf{V}_B and \mathbf{V}_A are also coplanar. Hence, the vector \mathbf{V}_B , like the vector \mathbf{V}_A , will be perpendicular to the vector $\boldsymbol{\omega}$. This allows us to make the following conclusion: if the vectors \mathbf{V}_A and $\boldsymbol{\omega}$ are mutually perpendicular with our choice of the origin of the frame K' , these vectors will also be mutually perpendicular at any other choice of the origin of the frame K' (with any other choice of the point A).

Let us now turn to formula (21.1) and write it as

$$\mathbf{V} = \mathbf{V}_A + [\boldsymbol{\omega}, \mathbf{r}_{(A)}] \quad (21.8)$$

This formula shows that when the vectors \mathbf{V}_A and $\boldsymbol{\omega}$ are mutually perpendicular (which, if this occurs, is observed with any choice of the point A), the vectors \mathbf{V} and \mathbf{V}_A will be coplanar, and the velocities \mathbf{V} of all the points of a body are in planes perpendicular to the vector $\boldsymbol{\omega}$. By varying our choice of the point A , we can find a position of it for which

$$\mathbf{V}_A = \mathbf{V} - [\boldsymbol{\omega}, \mathbf{r}_{(A)}] \quad (21.9)$$

vanishes¹ (here the point A may be outside the body). As a result,

¹ Both terms on the right-hand side of (21.9) are position functions of points of the body (\mathbf{V} is the velocity of a point of the body in the frame K , and $\mathbf{r}_{(A)}$ is the position vector of this point in the frame K'). The difference of these terms for all the points of the body is the same and equals \mathbf{V}_A .

the motion of a solid will be represented as only rotation about an axis called the **instantaneous axis of rotation** of the body [see (21.8)].

When the vectors \mathbf{V}_A and $\boldsymbol{\omega}$ are not perpendicular to each other, we can choose the point A so that these vectors will be collinear. Consequently, the motion of the body at each instant is the superposition of two motions: rotation about an axis at the angular velocity $\boldsymbol{\omega}$ and translational motion at the velocity \mathbf{V}_A along the same axis. We shall not stop to prove this statement.

We must note that the formulas of the dynamics of a rigid body become especially simple if we make the origin of the frame K' coincide with the centre of mass C of a body. In the following, we shall usually proceed in exactly this way. Formula (21.8) will therefore acquire the form

$$\mathbf{V} = \mathbf{V}_C + [\boldsymbol{\omega} \mathbf{r}] \quad (21.10)$$

22. The Euler Angles

The Euler angles are determined as follows. Assume that the axes of the frame K' associated with a body first coincided with the axes of the frame K . Next the body turned, as a result of which the

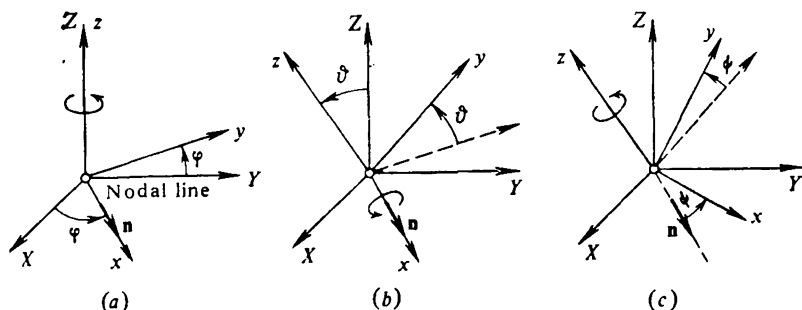


Fig. 22.1.

orientation of the axes of K' in space changed. Any such rotation can be performed with the aid of the three rotations shown in Fig. 22.1.

1. Rotation about the Z -axis through the angle φ (Fig. 22.1a). The direction \mathbf{n} followed by the x -axis is called the **nodal line**.
2. Rotation about the nodal line through the angle ϑ (Fig. 22.1b).
3. Rotation about the z -axis through the angle ψ (Fig. 22.1c).

The direction of each of these rotations is related to the direction of the axis about which it occurs by the right-hand screw rule.

Examination of Fig. 22.2 shows that the nodal line is the line of intersection of the coordinate planes XY and xy . The angle φ is formed

by the X -axis and the nodal line, the angle ψ by the nodal line and the x -axis, and, finally, the angle ϑ is the angle between the axes Z

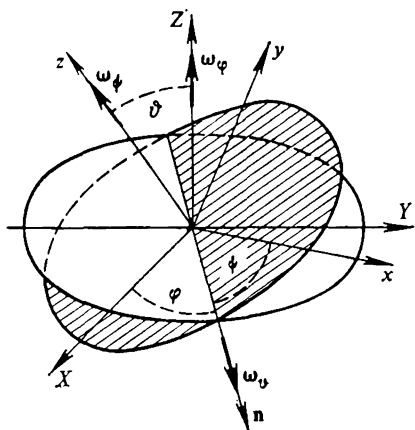


Fig. 22.2.

and z . The angles ϑ and φ are the polar coordinates of the point of intersection of the z -axis and a sphere of unit radius. This point is known as the apex.

For the set of angles φ , ϑ , and ψ determining each real rotation to be unique, it is assumed that the angles φ and ψ can have values from zero to 2π , while the values of the angle ϑ are limited to the interval from zero to π . If the angle ϑ were also allowed to have values from 0 to 2π , the rotation depicted in Fig. 22.3, for instance, could be characterized either by the set of angles $\varphi = \pi/2$, $\vartheta = \pi/2$, $\psi = 0$ (the upper sequence of rotations; the

axes X , Y , Z are not shown in the figure, their orientation coincides with the initial orientation of the axes x , y , z) or by the set $\varphi = 3\pi/2$, $\vartheta = 3\pi/2$, $\psi = \pi$ (the lower sequence of rotations).

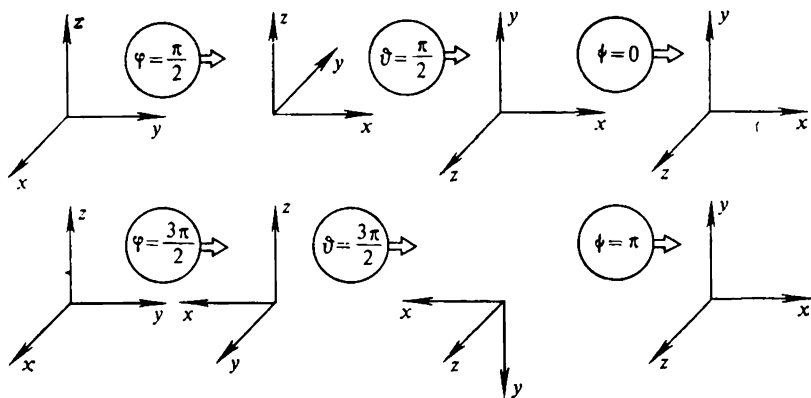


Fig. 22.3.

Assume that the Z -axis is directed vertically and the frame K' is rigidly associated with a top (gyroscope), the z -axis coinciding with the top's axis of proper rotation. It is now a simple matter to see that a change in the angle ψ corresponds to rotation of the top

itself, a change in the angle φ corresponds to rotation of the vertical plane containing the z -axis, i.e. to precession of the top, and, finally, a change in the angle ϑ to motion of the top's axis called nutation¹. Accordingly, the angle φ is called the precession angle, the angle ϑ —the nutation angle, and the angle ψ —the angle of proper rotation (or the angle of pure rotation)².

The rate of the change in the angle φ can be characterized by the angular velocity vector ω_φ directed along the Z -axis (see Fig. 22.2); the magnitude of this vector is $\dot{\varphi}$. Let us resolve the vector ω_φ into two components, one of which is directed along the z -axis (its magnitude is $\dot{\varphi} \cos \vartheta$), and the second is perpendicular to the z -axis, i.e. is in the plane xy (its magnitude is $\dot{\varphi} \sin \vartheta$). The second component is obviously perpendicular to the nodal line n and, consequently, makes the angles $\pi/2 - \psi$ and ψ with the axes x and y , respectively. We can conclude from the above that the projections of the vector ω_φ onto the axes of the frame K' are

$$\left. \begin{aligned} (\omega_\varphi)_1 &= \dot{\varphi} \sin \vartheta \cos (\pi/2 - \psi) = \dot{\varphi} \sin \vartheta \sin \psi \\ (\omega_\varphi)_2 &= \dot{\varphi} \sin \vartheta \cos \psi \\ (\omega_\varphi)_3 &= \dot{\varphi} \cos \vartheta \end{aligned} \right\} \quad (22.1)$$

The rate of the change in the angle ϑ is characterized by the vector ω_ϑ directed along the nodal line: its magnitude is $\dot{\vartheta}$. The nodal line is perpendicular to the z -axis, and makes the angles ψ and $\psi + \pi/2$ with the axes x and y , respectively. Consequently, the projections of the vector ω_ϑ onto the axes of the frame K' are

$$\left. \begin{aligned} (\omega_\vartheta)_1 &= \dot{\vartheta} \cos \psi \\ (\omega_\vartheta)_2 &= \dot{\vartheta} \cos (\psi + \pi/2) = -\dot{\vartheta} \sin \psi \\ (\omega_\vartheta)_3 &= 0 \end{aligned} \right\} \quad (22.2)$$

Finally, the rate of the change in the angle ψ is characterized by the vector ω_ψ directed along the z -axis (its magnitude is $\dot{\psi}$). The projections of this vector onto the axes of the frame K' are

$$(\omega_\psi)_1 = 0, \quad (\omega_\psi)_2 = 0, \quad (\omega_\psi)_3 = \dot{\psi} \quad (22.3)$$

¹ General courses of physics usually deal only with regular precession characterized by the angle between the top's axis and a vertical line remaining unchanged. Actually, as a rule, the top's axis oscillates in the plane Zz about a certain middle position. This oscillation is what we call nutation.

² The letter ψ is sometimes used to denote the angle of precession, and the letter φ —the angle of proper rotation.

The vector of the angular velocity ω at which the body rotates relative to the frame K (the body is always stationary relative to the frame K') can be represented as the sum of the angular velocities of each of the three rotations corresponding to the changes in the Euler angles:

$$\omega = \omega_\varphi + \omega_\vartheta + \omega_\psi$$

Therefore, the following values are obtained for the projections of the angular velocity ω onto the axes of the frame K' with account taken of formulas (22.1), (22.2), and (22.3):

$$\left. \begin{aligned} \omega_1 &= \dot{\varphi} \sin \vartheta \sin \psi + \dot{\vartheta} \cos \varphi \\ \omega_2 &= \dot{\varphi} \sin \vartheta \cos \psi - \dot{\vartheta} \sin \varphi \\ \omega_3 &= \dot{\varphi} \cos \vartheta + \dot{\psi} \end{aligned} \right\} \quad (22.4)$$

We shall need these formulas in the following.

23. The Inertia Tensor ¹

Assume that we are observing the motion of a solid in the stationary reference frame K (whose axes will be designated X_1, X_2, X_3 , or X, Y, Z). In accordance with what was stated in Sec. 21, let us connect the reference frame K' having the axes x_1, x_2, x_3 (or x, y, z) to the body rigidly. We divide the body mentally into particles of mass m_α ². According to formula (21.10), the velocity of the α -th particle will be written as follows:

$$\mathbf{V}_\alpha = \mathbf{V}_C + [\omega, \mathbf{r}_\alpha] \quad (23.1)$$

Let us calculate the kinetic energy of the body. It is

$$\begin{aligned} T &= \frac{1}{2} \sum_\alpha m_\alpha \mathbf{V}_\alpha^2 = \frac{1}{2} \sum_\alpha m_\alpha \{ \mathbf{V}_C + [\omega, \mathbf{r}_\alpha] \}^2 \\ &= \frac{1}{2} \sum_\alpha m_\alpha \mathbf{V}_C^2 + \sum_\alpha m_\alpha \mathbf{V}_C [\omega, \mathbf{r}_\alpha] + \frac{1}{2} \sum_\alpha m_\alpha [\omega, \mathbf{r}_\alpha]^2 \end{aligned}$$

In the first term on the right-hand side, we can put the factor \mathbf{V}_C^2 outside the sum sign. This term therefore becomes $\frac{1}{2} m \mathbf{V}_C^2$, where $m = \sum m_\alpha$ is the mass of the body. In the second term, we shall perform a cyclic transposition of the factors [see (VI.3)], after which we shall put the constant factor outside the sum sign. The result is the expression $[\mathbf{V}_C, \omega] \sum m_\alpha \mathbf{r}_\alpha = [\mathbf{V}_C, \omega] m \mathbf{r}_C$, where \mathbf{r}_C is the

¹ Before beginning to study this section, acquaint yourself with Appendix X.

² The subscript α indicates a particle's number. We use the Latin subscripts i, k, l, \dots to number coordinate axes, vector components, etc.

position vector of the centre of mass C . If, as we have agreed upon, we place the origin of the frame K' at the point C , the second term vanishes.

Hence, the kinetic energy of a rigid body breaks up into two terms. The first of them,

$$T_{\text{trans}} = \frac{1}{2} m V_C^2 \quad (23.2)$$

is the kinetic energy of translational motion. The second term

$$T_{\text{rot}} = \frac{1}{2} \sum_{\alpha} m_{\alpha} [\omega, \mathbf{r}_{\alpha}]^2 \quad (23.3)$$

is the kinetic energy of rotation.

We must stress the fact that both these energies are absolutely independent—one depends only on V_C , and the other only on ω . Since the origin of the frame K' coincides with the point C , the term containing both V_C and ω vanishes.

Let us transform expression (23.3). First, we shall replace the square of the vector product in accordance with (VI.6):

$$T_{\text{rot}} = \frac{1}{2} \sum_{\alpha} m_{\alpha} \{ \omega^2 \mathbf{r}_{\alpha}^2 - (\omega \mathbf{r}_{\alpha})^2 \} \quad (23.4)$$

We shall now write this expression using the projections of the vectors ω and \mathbf{r}_{α} onto the axes of the frame K' . The projections of \mathbf{r}_{α} onto the axes of the frame K' equal simply the coordinates of the particle $x_{1\alpha}, x_{2\alpha}, x_{3\alpha}$. Let $\omega_1, \omega_2, \omega_3$ stand for the projections of the vector ω onto the axes of the frame K' . Expression (23.4) therefore acquires the following form in the components of the vectors:

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2} \sum_{\alpha} m_{\alpha} \left[\left(\sum_i \omega_i^2 \right) \left(\sum_l x_{l\alpha}^2 \right) - \left(\sum_i \omega_i x_{i\alpha} \right) \left(\sum_k \omega_k x_{k\alpha} \right) \right] \\ &= \frac{1}{2} \sum_{\alpha} m_{\alpha} \left[\left(\sum_i \omega_i^2 \right) \left(\sum_l x_{l\alpha}^2 \right) - \sum_{i,k} \omega_i \omega_k x_{i\alpha} x_{k\alpha} \right] \end{aligned}$$

(we remind our reader that a dummy index can be designated by any letter).

In the expression we have obtained, the quantities ω_i and ω_k do not depend on the subscript α and they could be put outside the sign of the sum over α . This is prevented by the circumstance, however, that the first term of the expression includes the sum of the squares ω_i^2 and the second term, the sum of the products $\omega_i \omega_k$. This obstacle can be eliminated by replacing the sum of the quantities ω_i^2 with the expression $\sum_{i,k} \omega_i \omega_k \delta_{ik}$ which is obviously equivalent

to $\sum_i \omega_i^2$. The formula for the rotational energy therefore becomes

$$T_{\text{rot}} = \frac{1}{2} \sum_{\alpha} m_{\alpha} \left[\left(\sum_{i, k} \omega_i \omega_k \delta_{ik} \right) \left(\sum_l x_{l\alpha}^2 \right) - \sum_{i, k} \omega_i \omega_k x_{i\alpha} x_{k\alpha} \right] \quad (23.5)$$

(we must note that $\sum_l x_{l\alpha}^2$ is simply a scalar¹ depending on the subscript α ; each of the addends $\omega_i \omega_k \delta_{ik}$ is multiplied by this scalar).

In expression (23.5), summation is first performed over the subscripts i and k , and then over the subscript α . Let us change the sequence of summation so that summation over the subscripts i and k will be performed the last, i.e. rewrite (23.5) as follows:

$$T_{\text{rot}} = \frac{1}{2} \sum_{i, k} \omega_i \omega_k \sum_{\alpha} m_{\alpha} \left[\delta_{ik} \left(\sum_l x_{l\alpha}^2 \right) - x_{i\alpha} x_{k\alpha} \right]$$

If we introduce the symbol²

$$I_{ik} = \sum_{\alpha} m_{\alpha} \left[\delta_{ik} \left(\sum_l x_{l\alpha}^2 \right) - x_{i\alpha} x_{k\alpha} \right] \quad (23.6)$$

the expression for the kinetic energy of rotation can be written as

$$T_{\text{rot}} = \frac{1}{2} \sum_{i, k} I_{ik} \omega_i \omega_k \quad (23.8)$$

The quantity determined by formula (23.6) is a number (but not an invariant!) depending on the subscripts i and k . There are altogether nine such numbers. It can be seen that the set of quantities I_{ik} forms a second-rank tensor³. Indeed, the product of the scalar

$$\sum_l x_{l\alpha}^2 = r_{\alpha}^2 = \text{inv.}$$

² When calculating the quantities I_{ik} for a continuous body, ρdV must be taken instead of m_{α} and summation replaced by integration. Hence

$$I_{ik} = \int \left[\delta_{ik} \left(\sum_l x_l^2 \right) - x_i x_k \right] \rho dV \quad (23.7)$$

³ This conclusion can also be arrived at by the following reasoning. Let us write expression (23.8) in the form

$$T_{\text{rot}} = \frac{1}{2} \sum_i \omega_i \sum_k I_{ik} \omega_k$$

This expression can be invariant only if the quantities $\sum_k I_{ik} \omega_k$ are the i -th components of a vector [see Appendix VI, the text following formula (VI.28)]. The latter, in turn, is possible only when the quantities I_{ik} are the components of a tensor [see Appendix X, the text between formulas (X.22) and (X.23)].

$\sum_i x_{i\alpha}^2$ and the unit tensor δ_{ik} is a tensor [see formulas (X.17) and (X.19)], and the products $x_{i\alpha}x_{k\alpha}$ are products of the components of the vector \mathbf{r}_α , i.e. also a tensor [see (X.16)]. Finally, the difference of the relevant components of two tensors also gives the components of a tensor [see (X.18)].

Hence, quantities (23.6) are the components of a tensor. The latter is called an **inertia tensor**. Quantities (23.6) do not change when the subscripts i and k are transposed. Consequently, the inertia tensor is symmetrical ($I_{ik} = I_{ki}$).

Let us write the components of the inertia tensor using the conventional notation of Cartesian coordinates:

$$(I_{ik}) = \begin{bmatrix} \sum m(y^2 + z^2) & -\sum mxy & -\sum mxz \\ -\sum myx & \sum m(x^2 + z^2) & -\sum myz \\ -\sum mzx & -\sum mzy & \sum m(x^2 + y^2) \end{bmatrix} \quad (23.9)$$

(to avoid making the formulas more complicated, we have suppressed the subscript α on m and on the coordinates x, y, z ; all the sums are taken over this subscript).

The diagonal components of the tensor are known as **axial moments of inertia**. They coincide with the moments of inertia of a body relative to the corresponding coordinate axes known from the general course of physics. The non-diagonal components are called **centrifugal moments of inertia**.

The geometric shape of a symmetric tensor is an ellipsoid. In the case being considered, it is an ellipsoid of inertia. The directions in a body coinciding with the semiaxes of an ellipsoid of inertia are called its **principal axes of inertia**. They intersect at the centre of mass of the body. If we direct the axes of the frame K' (i.e. the axes x, y, z ; we remind our reader that these axes are rigidly connected to the body) along the principal axes of inertia of a body, the inertia tensor will be reduced to a diagonal form

$$(I_{ik}) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad (23.10)$$

The values of I_1, I_2, I_3 of the diagonal components of a tensor (in the case when it] has been reduced to a diagonal form) are called the] **principal moments of inertia** of a body (they could be designated by the symbols I_x, I_y, I_z).

If the principal axes of inertia have been chosen as the axes x, y, z associated with a body, expression (23.8) for the kinetic energy of rotation is simplified as follows:

$$T_{\text{rot}} = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \quad (23.11)$$

or

$$T_{\text{rot}} = \frac{1}{2} (I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2) \quad (23.12)$$

(do not forget that $\omega_1, \omega_2, \omega_3$ are the projections onto the axes x, y, z of the vector ω —the angular velocity of rotation of the body observed in the frame with the axes X, Y, Z).

When the vector ω coincides with one of the principal axes of inertia along which we direct, say, the z -axis, the expression for the energy becomes even more simple:

$$T_{\text{rot}} = \frac{1}{2} I_z \omega^2 \quad (23.13)$$

An expression similar to (23.13) is also obtained when a body rotates about an axis fixed relative to the body and passing through its centre of mass¹. By directing, say, the z -axis along this axis, we find that $\omega_x = \omega_y = 0$, and $\omega_z = \omega$. Consequently, of the nine addends of formula (23.8), only the one in which $i = k = z$ will be non-zero, so that

$$T_{\text{rot}} = \frac{1}{2} I_{zz} \omega_z^2 = \frac{1}{2} I_{zz} \omega^2 \quad (23.14)$$

where I_{zz} in the general case is not one of the principal moments of inertia.

We must note that, for example, for a body such as a homogeneous sphere, the ellipsoid of inertia degenerates into a sphere. Therefore, the principal axes of inertia are not fixed relative to the body. This signifies that any three mutually perpendicular axes passing through the centre of the sphere can be taken as the principal axes. In this case, all the principal moments of inertia are the same: $I_1 = I_2 = I_3 = I$, and the tensor of inertia can be written as

$$(I_{ik}) = I (\delta_{ik}) \quad (23.15)$$

where (δ_{ik}) is a unit tensor [see (X.17)], and I is a scalar.

Everything that we have said about a sphere also holds for a homogeneous cube. Indeed, relation (23.15) evidently holds for it. Consequently, the ellipsoid of inertia for a cube degenerates into a sphere. For this reason, any axis (and not only an axis of symmetry) passing through the centre of a cube may be considered as a principal axis of inertia. This is why a cube, in addition to a sphere and other bodies for which $I_1 = I_2 = I_3$, is called a spherical top².

¹ If this axis does not coincide with any of the principal axes of inertia, it must be retained in place with the aid of bearings.

² For a spherical top, the energy is always expressed by formula (23.13), where by I_z we must understand the scalar factor I in expression (23.15) for the inertia tensor.

A body for which two principal moments of inertia are equal (for instance, $I_1 = I_2 \neq I_3$) is called a symmetrical top. Finally, a body for which all three principal moments of inertia are different is called an asymmetrical top.

Up to now, in considering the inertia tensor, we assumed that the origin of the frame K' associated with a body is at the centre of mass C of the body. The inertia tensor can also be determined by formula (23.6), however, with respect to the frame $K'_{(A)}$ associated with a body and having its origin at an arbitrary point A . The tensor components in this case will be

$$I_{ih}^{(A)} = \sum_{\alpha} m_{\alpha} \{ \delta_{ih} [\sum_l (x_{l\alpha}^{(A)})^2] - x_{i\alpha}^{(A)} x_{h\alpha}^{(A)} \} \quad (23.16)$$

The values of $I_{ih}^{(A)}$ are related by simple expressions to the values I_{ih} of the tensor components determined with respect to the frame $K'_{(C)}$ with its origin at the point C and with axes parallel to those of the frame $K'_{(A)}$ (Fig. 23.1). To find these relations, let us use the symbol \mathbf{a} to designate the position vector of the point A in the frame $K'_{(C)}$. Hence, for any point of the body, $\mathbf{r}_{(A)} = \mathbf{r} - \mathbf{a}$ and, consequently,

$$x_i^{(A)} = x_i - a_i \quad (23.17)$$

where a_i is the i -th coordinate of the point A in the frame $K'_{(C)}$.

Let us introduce the values (23.17) into formula (23.16):

$$\begin{aligned} I_{ih}^{(A)} &= \sum_{\alpha} m_{\alpha} \{ \delta_{ih} [\sum_l (x_{l\alpha} - a_l)^2] - (x_{i\alpha} - a_i) (x_{h\alpha} - a_h) \} \\ &= \sum_{\alpha} m_{\alpha} \{ \delta_{ih} (\sum_l x_{l\alpha}^2) - x_{i\alpha} x_{h\alpha} \} + \sum_{\alpha} m_{\alpha} \{ \delta_{ih} (\sum_l a_l^2) - a_i a_h \} \\ &\quad - \sum_{\alpha} m_{\alpha} \delta_{ih} \sum_l 2x_{l\alpha} a_l + \sum_{\alpha} m_{\alpha} x_{i\alpha} a_h + \sum_{\alpha} m_{\alpha} x_{h\alpha} a_i \end{aligned}$$

The first of the five sums on the right-hand side is I_{ih} . In the second sum, none of the quantities in the braces has the subscript α . In addition, $\sum_l a_l^2$ is the square of the vector \mathbf{a} , i.e. a^2 . Therefore, the second sum can be written as $m(a^2 \delta_{ih} - a_i a_h)$. The third sum can be written as $\sum_l 2\delta_{ih} a_l \sum_{\alpha} m_{\alpha} x_{l\alpha}$. But $\sum_{\alpha} m_{\alpha} x_{l\alpha} = m x_{lC} = 0$, so that the third sum vanishes. Similarly, factoring out the multiplier in

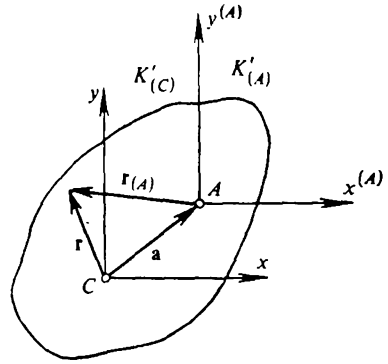


Fig. 23.1.

the fourth and fifth sums that does not depend on α and taking into account that $x_{iC} = x_{hC} = 0$, we find that both these sums also vanish. We thus arrive at the relation

$$I_{ih}^{(A)} = I_{ih} + m(a^2\delta_{ih} - a_i a_h) \quad (23.18)$$

which is the tensor form of writing the parallel axis (Steiner) theorem. To verify this, let us find the component $I_{33}^{(A)} = I_{zz}^{(A)}$. According to (23.18)

$$I_{zz}^{(A)} = I_{zz} + m(a^2 - a_z^2) = I_{zz} + m(a_x^2 + a_y^2) = I_{zz} + ma_{\perp}^2 \quad (23.19)$$

where a_{\perp} is the distance between the axes z and $z^{(A)}$.

We calculate the kinetic energy of a body rotating with the angular velocity ω about a stationary (in the frame K) axis fixed relative to the body and not passing through its centre of mass. If the centre of mass is at a distance a_{\perp} from the axis of rotation, its velocity is $V_C = \omega a_{\perp}$. Consequently,

$$T_{\text{trans}} = \frac{1}{2} m \omega^2 a_{\perp}^2$$

where m is the mass of the body. We find the rotational energy by formula (23.14), directing the z -axis parallel to the stationary axis of rotation. Summating both energies, we obtain

$$T = \frac{1}{2} m \omega^2 a_{\perp}^2 + \frac{1}{2} I_{zz} \omega^2 = \frac{1}{2} (ma_{\perp}^2 + I_{zz}) \omega^2 = \frac{1}{2} I_{zz}^{(A)} \omega^2 \quad (23.20)$$

[see formula (23.19)].

Consequently, the formula for the kinetic energy of a rotating body considered in the general course of physics is true not always, but only in two cases: (1) when the body rotates about one of its principal axes of inertia [see formula (23.13)], and (2) when the body rotates about a stationary axis fixed in it [see formulas (23.14) and (23.20)].

In concluding, we shall find the form of the inertia tensor when only one of the coordinate axes, say the z -axis, coincides with one of the principal axes of inertia of a body. The transition from a frame all of whose axes coincide with the principal ones to the frame we are interested in is achieved by rotation about the z -axis through the angle φ . It is easy to see that the table of the transformation coefficients is as follows in this case:

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

The components of the inertia tensor in the reference frame we are interested in are obtained from the components of the tensor

reduced to the principal axes according to the transformation formula (X.10). Let us represent the components of the tensor (23.10) in the form $I_{lm} = \delta_{lm}I_m$. Therefore, denoting the components of the tensor in the new "primed" frame simply by I_{ik} (without a prime), we can write

$$I_{ik} = \sum_{l, m} \alpha_{il} \alpha_{km} \delta_{lm} I_m = \sum_m \alpha_{im} \alpha_{km} I_m$$

According to this formula

$$I_{11} = \sum_m \alpha_{1m}^2 I_m = \alpha_{11}^2 I_1 + \alpha_{12}^2 I_2 = \cos^2 \varphi I_1 + \sin^2 \varphi I_2$$

Similar calculations show that $I_{22} = \sin^2 \varphi I_1 + \cos^2 \varphi I_2$, $I_{33} = I_3$, $I_{12} = I_{21} = \sin \varphi \cos \varphi (I_2 - I_1)$, $I_{13} = I_{31} = 0$, $I_{23} = I_{32} = 0$. Hence, the inertia tensor in the new frame has the form

$$(I_{ik}) = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad (23.21)$$

We must note that when $I_1 = I_2 = I$, it follows from the formulas we have obtained that $I_{11} = I_{22} = I$, and $I_{12} = I_{21} = 0$, i.e. that the new tensor does not differ from the initial one. This is exactly how matters should stand because when the moments I_1 and I_2 are equal, the principal axes x and y are not fixed.

24. Angular Momentum of a Rigid Body

As in the preceding sections, we shall consider the motion of a rigid body in the reference frame K with the axes X_1, X_2, X_3 . We shall rigidly associate with the body the frame K' whose origin will first be placed at an arbitrary point A . We shall designate the axes of this frame by x_1, x_2, x_3 (or x, y, z). We shall mentally divide the body into particles of mass m_α .

By formula (21.8), the velocity of the α -th particle in the frame K is

$$\mathbf{V}_\alpha = \mathbf{V}_A + [\boldsymbol{\omega}, \mathbf{r}_\alpha^{(A)}] \quad (24.1)$$

where \mathbf{V}_A is the velocity of the origin of the frame K' , $\boldsymbol{\omega}$ is the angular velocity of rotation of the body in the frame K , and $\mathbf{r}_\alpha^{(A)}$ is the position vector of the particle emerging from the point A .

Let us find the angular momentum $\mathbf{M}_{(A)}$ of the body relative to the origin of the frame K' (relative to the point A). The position vector leading from the point A to the α -th particle is $\mathbf{r}_\alpha^{(A)}$. Hence,

$$\mathbf{M}_{(A)} = \sum_\alpha [\mathbf{r}_\alpha^{(A)}, m_\alpha \mathbf{V}_\alpha]$$

Substitution into this expression of the value (24.1) for V_α yields

$$M_{(A)} = \sum_{\alpha} [r_{\alpha}^{(A)}, m_{\alpha} V_A] + \sum_{\alpha} [r_{\alpha}^{(A)}, m_{\alpha} [\omega, r_{\alpha}^{(A)}]] \quad (24.2)$$

The second term in (24.2) is the value $M'_{(A)}$ which the angular momentum would have provided that the point A were stationary. Consequently, $M'_{(A)}$ is the angular momentum due only to rotation of a body.

Let us transform the first term in (24.2) using the distributivity of a vector product:

$$\begin{aligned} \sum_{\alpha} [r_{\alpha}^{(A)}, m_{\alpha} V_A] &= \sum_{\alpha} [m_{\alpha} r_{\alpha}^{(A)}, V_A] = [(\sum_{\alpha} m_{\alpha} r_{\alpha}^{(A)}), V_A] \\ &= [m r_C^{(A)}, V_A] = [r_C^{(A)}, m V_A] \end{aligned}$$

Here m is the mass of the body, and $r_C^{(A)}$ is the position vector of the centre of mass in the frame K' (the position vector from A to C).

Expression (24.2) can thus be given the form

$$M_{(A)} = M'_{(A)} + [r_C^{(A)}, m V_A] \quad (24.3)$$

We have already noted that the term $M'_{(A)}$ is due to rotation of the body. It can be called the proper angular momentum of the body. The second term is due to the translational motion of the body.

If we place the origin of the frame K' (i.e. the point A) at the centre of mass of the body, $r_C^{(A)}$ vanishes, and as a result (24.3) becomes $M_{(C)} = M'_{(C)}$. It thus follows that the angular momentum of a body relative to its centre of mass observed in the stationary frame K coincides with its proper angular momentum, i.e. it is determined only by rotation and does not depend on whether the centre of mass of the body is moving or is at rest.

Let us find an expression for the angular momentum¹ of a body relative to its centre of mass. If the point A coincides with C , the first term in (24.2) vanishes. Consequently,

$$M_{(C)} = \sum_{\alpha} m_{\alpha} [r_{\alpha}, [\omega, r_{\alpha}]] \quad (24.4)$$

(recall that r_{α} is the position vector of a particle emerging from point C). Let us transform this expression according to formula (VI.5):

$$M_{(C)} = \sum_{\alpha} m_{\alpha} \{ \omega r_{\alpha}^2 - r_{\alpha} (\omega r_{\alpha}) \} = \sum_{\alpha} m_{\alpha} \{ \omega (\sum_l x_{l\alpha}^2) - r_{\alpha} (\sum_h \omega_h x_{h\alpha}) \}$$

(we have expressed the scalar products in terms of projections of the relevant vectors onto the axes of the frame K' associated with the body).

¹ Observed in the frame K . In the frame K' , the body is at rest, so that the angular momentum in this frame is always zero.

Let us calculate the components of the vector $\mathbf{M}_{(C)}$ along the axes of the frame K' (in the following formulas we shall omit the subscript "C" in parentheses). For the component along the i -th axis, we obtain

$$M_i = \sum_{\alpha} m_{\alpha} \left\{ \omega_i \left(\sum_l x_{l\alpha}^2 \right) - x_{i\alpha} \left(\sum_h \omega_h x_{h\alpha} \right) \right\}$$

Let us write ω_i in the form

$$\omega_i = \sum_h \delta_{ih} \omega_h$$

Hence

$$M_i = \sum_{\alpha} m_{\alpha} \left\{ \sum_h \delta_{ih} \omega_h \left(\sum_l x_{l\alpha}^2 \right) - x_{i\alpha} \sum_h \omega_h x_{h\alpha} \right\}$$

Finally, let us change the sequence of summation over the subscripts α and k :

$$M_i = \sum_h \omega_h \left[\sum_{\alpha} m_{\alpha} \left\{ \delta_{ih} \left(\sum_l x_{l\alpha}^2 \right) - x_{i\alpha} x_{h\alpha} \right\} \right]$$

The expression in brackets is a component of the inertia tensor I_{ih} [see formula (23.6)]. Consequently, for the projection of the vector \mathbf{M} onto the i -th axis of the reference frame associated with the body, we get the following expression

$$M_i = \sum_h I_{ih} \omega_h \quad (i = 1, 2, 3) \quad (24.5)$$

(do not forget that ω_h is the projection of the vector $\boldsymbol{\omega}$ onto the h -th axis of the frame K').

Examination of formula (24.5) shows that the vectors \mathbf{M} and $\boldsymbol{\omega}$ are, in general, not collinear to each other. If the axes of the frame K' (i.e. the axes x, y, z) are directed along the principal axes of inertia of a body, formula (24.5) becomes simplified as follows:

$$M_i = I_i \omega_i \quad (i = x, y, z) \quad (24.6)$$

Here I_i is the i -th principal moment of inertia.

Assume that a body rotates, for example, about the third principal axis of inertia. Therefore, $\omega_x = \omega_y = 0$, and $\omega_z = \omega$, so that

$$M = M_z = I_z \omega_z = I_z \omega$$

The last relation can be written in the vector form:

$$\mathbf{M} = I_z \boldsymbol{\omega} \quad (24.7)$$

A glance at formula (24.6) shows that for a spherical top (i.e. a body for which $I_1 = I_2 = I_3 = I$), the vectors \mathbf{M} and $\boldsymbol{\omega}$ will also be collinear, and $\mathbf{M} = I\boldsymbol{\omega}$.

If a body rotates about the z -axis fixed in it that does not coincide with any of the principal axes of inertia, $\omega_x = \omega_y = 0$, and $\omega_z = \omega$.

Formula (24.5) therefore yields

$$M_x = I_{xz}\omega, \quad M_y = I_{yz}\omega, \quad M_z = I_{zz}\omega$$

The projection of the vector \mathbf{M} onto an axis passing through the point relative to which \mathbf{M} has been determined is called the angular momentum of the body relative to this axis. Consequently, the angular momentum relative, for example, to the z -axis is

$$M_z = I_{zz}\omega \quad (24.8)$$

We must stress that the vector \mathbf{M} itself in the last case is not collinear to the vector ω and rotates about the direction of ω together with the axes x and y .

In conclusion, let us consider the case when a body rotates about a stationary (in the frame K) axis fixed in it that does not pass through its centre of mass C (Fig. 24.1). Let us take an arbitrary point A on the axis of rotation. We draw a vector which we shall designate by the symbol \mathbf{a} from

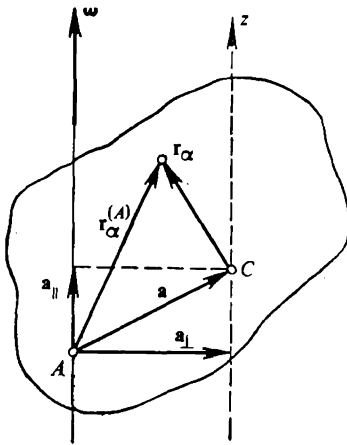


Fig. 24.1.

this point to the point C . Therefore, for each of the body's particles

$$\mathbf{r}_\alpha^{(A)} = \mathbf{a} + \mathbf{r}_\alpha \quad (24.9)$$

where $\mathbf{r}_\alpha^{(A)}$ is the position vector of the α -th particle emerging from the point A , and \mathbf{r}_α is the position vector of the same particle from the point C .

According to our assumption, the axis of rotation is stationary. Consequently, the velocity \mathbf{V}_A equals zero. Let us find the angular momentum of the body relative to the point A , i.e. $\mathbf{M}_{(A)}$. By formula (24.2)

$$\mathbf{M}_{(A)} = \sum_{\alpha} [\mathbf{r}_\alpha^{(A)}, m_{\alpha} [\omega, \mathbf{r}_\alpha^{(A)}]]$$

We introduce into this expression the value of $\mathbf{r}_\alpha^{(A)}$ from (24.9):

$$\mathbf{M}_{(A)} = \sum_{\alpha} [(\mathbf{a} + \mathbf{r}_\alpha), m_{\alpha} [\omega, (\mathbf{a} + \mathbf{r}_\alpha)]] \quad (24.10)$$

Taking advantage of the distributivity of a vector product, after simple transformations we can write expression (24.10) as follows:

$$\begin{aligned} \mathbf{M}_{(A)} = & [\mathbf{a}, [\omega \mathbf{a}]] \left(\sum_{\alpha} m_{\alpha} \right) + [\mathbf{a}, [\omega, (\sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha})]] \\ & + [(\sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha}), [\omega \mathbf{a}]] + \sum_{\alpha} [\mathbf{r}_{\alpha}, m_{\alpha} [\omega \mathbf{r}_{\alpha}]] \end{aligned} \quad (24.11)$$

Since $\sum m_\alpha \mathbf{r}_\alpha = m \mathbf{r}_C = 0$, the second and third terms in the expression we have obtained vanish. Using formula (VI.5) and substituting the mass m of the body for $\sum m_\alpha$, we transform the first term into

$$m [\omega a^2 - \mathbf{a} (\omega \mathbf{a})] = m [(a_{\parallel}^2 + a_{\perp}^2) \omega - a_{\parallel} \omega \mathbf{a}],$$

where a_{\parallel} is the component of the vector \mathbf{a} parallel to ω , and a_{\perp} is the component of \mathbf{a} perpendicular to ω (a_{\perp} is the distance between the axis of rotation and the z -axis parallel to it and passing through the point C).

Finally, a comparison with formula (24.4) shows that the last term in (24.11) is $M_{(C)}$ —the angular momentum of the body relative to the point C . Hence, in the case shown in Fig. 24.1

$$M_{(A)} = m (a_{\parallel}^2 + a_{\perp}^2) \omega - m a_{\parallel} \omega \mathbf{a} + M_{(C)} \quad (24.12)$$

Let us find the component of the angular momentum (24.12) along the axis of rotation. We shall designate it $M_{\parallel}^{(A)}$. The component of the first term equals this term itself. Since $\omega a_{\parallel} = a_{\parallel} \omega$, the component of the second term can be written as $m a_{\parallel}^2 \omega$. By formula (24.8), the component of the third term is $I_{zz} \omega$, where I_{zz} is the moment of inertia of the body relative to the z -axis (passing through the point C). Consequently,

$$M_{\parallel}^{(A)} = m a_{\perp}^2 \omega + I_{zz} \omega = (m a_{\perp}^2 + I_{zz}) \omega = I_{zz}^{(A)} \omega \quad (24.13)$$

Here $I_{zz}^{(A)}$ is the moment of inertia of the body relative to a fixed axis of rotation passing through the point A determined by the parallel axis theorem [see formula (23.19)].

We must note that the vector $M_{(A)}$ itself, in general, does not coincide in direction with the vector ω .

25. Free Axes of Rotation

Free axes of rotation of a body are defined to be axes that retain their position in space without the action of external forces on them. We shall prove in the present section that only the principal axes of inertia can be free axes of rotation.

Assume that a body rotates at the angular velocity ω about a fixed axis associated with it (Fig. 25.1). The following acceleration must be imparted to each particle of the body:

$$\mathbf{w}_\alpha = -\omega^2 \rho_\alpha$$

where ρ_α is the component of the position vector \mathbf{r}_α of the given particle perpendicular to the axis of rotation

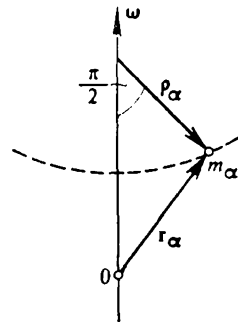


Fig. 25.1.

(\mathbf{r}_α emerges from the point O on the axis of rotation¹).

To impart such an acceleration to a particle, the force

$$\mathbf{F}_\alpha = m_\alpha \mathbf{w}_\alpha = -m_\alpha \omega^2 \boldsymbol{\rho}_\alpha \quad (25.1)$$

must be applied to it whose moment relative to the point O is

$$\mathbf{N}_\alpha = [\mathbf{r}_\alpha, \mathbf{F}_\alpha] = -m_\alpha \omega^2 [\mathbf{r}_\alpha, \boldsymbol{\rho}_\alpha] \quad (25.2)$$

By summing all the forces (25.1), we get the resultant external force that has to be applied to a body to ensure its rotation about the axis being considered²:

$$\mathbf{F} = \sum_\alpha \mathbf{F}_\alpha = -\omega^2 \sum_\alpha m_\alpha \boldsymbol{\rho}_\alpha \quad (25.3)$$

The resultant moment of the external forces must equal the sum of the moments (25.2):

$$\mathbf{N} = \sum_\alpha \mathbf{N}_\alpha = -\omega^2 \sum_\alpha m_\alpha [\mathbf{r}_\alpha, \boldsymbol{\rho}_\alpha] \quad (25.4)$$

Let us associate with a body a system of coordinates having its origin at the point O and its z -axis directed along the vector $\boldsymbol{\omega}$. The components of the vector $\boldsymbol{\rho}_\alpha$ along the axes of such a system are $x_\alpha, y_\alpha, 0$. Consequently,

$$[\mathbf{r}_\alpha, \boldsymbol{\rho}_\alpha] = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x_\alpha & y_\alpha & z_\alpha \\ x_\alpha & y_\alpha & 0 \end{vmatrix}$$

so that the components of the vector $[\mathbf{r}_\alpha, \boldsymbol{\rho}_\alpha]$ are

$$\begin{aligned} [\mathbf{r}_\alpha, \boldsymbol{\rho}_\alpha]_{\text{pr.}x} &= y_\alpha \cdot 0 - z_\alpha y_\alpha = -y_\alpha z_\alpha \\ [\mathbf{r}_\alpha, \boldsymbol{\rho}_\alpha]_{\text{pr.}y} &= z_\alpha x_\alpha - x_\alpha \cdot 0 = x_\alpha z_\alpha \\ [\mathbf{r}_\alpha, \boldsymbol{\rho}_\alpha]_{\text{pr.}z} &= x_\alpha y_\alpha - y_\alpha x_\alpha = 0 \end{aligned} \quad (25.5)$$

Now let us write the components of the resultant force \mathbf{F} and the resultant moment \mathbf{N} . By formula (25.3), we have

$$F_x = -\omega^2 \sum_\alpha m_\alpha x_\alpha = -\omega^2 m x_C$$

$$F_y = -\omega^2 \sum_\alpha m_\alpha y_\alpha = -\omega^2 m y_C$$

$$F_z = 0$$

¹ In accordance with the notation we have adopted, this vector ought to be designated by the symbol $\mathbf{r}_\alpha^{(0)}$. But since in this section we shall not encounter position vectors emerging from the point C , we shall suppress the index "(0)" on the symbols of vectors and coordinates to simplify our notation.

² The forces \mathbf{F}_α include both external and internal forces, but it is general knowledge that the resultant of the internal forces is zero.

where x_C and y_C are the coordinates of the body's centre of mass. If the axis of rotation passes through the centre of mass, these coordinates will be zero, so that all the components of the force and, consequently, the force F itself, will vanish.

By formulas (25.4) and (25.5), we obtain

$$N_x = -\omega^2 \sum_{\alpha} m_{\alpha} (-y_{\alpha} z_{\alpha}) = -\omega^2 I_{yz}$$

$$N_y = -\omega^2 \sum_{\alpha} m_{\alpha} (x_{\alpha} z_{\alpha}) = \omega^2 I_{xz}$$

$$N_z = -\omega^2 \sum_{\alpha} m_{\alpha} \cdot 0 = 0$$

where I_{yz} and I_{xz} are the centrifugal moments of inertia of the body [see formula (23.9)]. If the z -axis about which rotation occurs is one of the principal axes of inertia, the centrifugal moments I_{xz} and I_{yz} are zero [see (23.21)] so that all the components of the resultant moment of the forces and, consequently, the moment N itself vanish.

We have thus proved that when a body rotates about one of its principal axes of inertia, the resultant of the external forces and the resultant moment of these forces equal zero. Hence, for such an axis of rotation to retain its position in space, no external forces are needed.

26. Equation of Motion of a Rigid Body

Let us take as the generalized coordinates determining the position of a body in the stationary frame K the Cartesian coordinates X_{1C} , X_{2C} , X_{3C} of the centre of mass (the position vector R_C corresponds to them) and the Euler angles φ , θ , ψ , and let us direct the axes of the frame K' associated with the body along its principal axes of inertia.

We established in Sec. 23 that the kinetic energy of a rigid body consists of the energy of translational motion (23.2) and that of rotation, which with our choice of the axes of the frame K' is determined by formula (23.14). Hence, for the Lagrangian of a rigid body, we can write the following expression:

$$L = \frac{1}{2} m V_C^2 + \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) - U \quad (26.1)$$

To obtain an expression for L in the generalized coordinates we have adopted, let us substitute \dot{R}_C for V_C and express the projections of the vector ω onto the axes of the frame K' in terms of the Euler

angles [see (22.4)]. The result is

$$L = \frac{1}{2} m \dot{\mathbf{R}}_C^2 + \frac{1}{2} \{ I_1 (\dot{\varphi} \sin \vartheta \sin \psi + \dot{\vartheta} \cos \psi)^2 + I_2 (\dot{\varphi} \sin \vartheta \cos \psi - \dot{\vartheta} \sin \psi)^2 + I_3 (\dot{\varphi} \cos \vartheta + \dot{\psi})^2 \} - U(\mathbf{R}_C, \varphi, \vartheta, \psi) \quad (26.2)$$

(remember that I_1, I_2, I_3 are the principal moments of inertia of the body).

Knowing the form of the function $U(\mathbf{R}_C, \varphi, \vartheta, \psi)$, we can compile Lagrange's equation and solve the relevant problem on the motion of a rigid body. Lagrange's equation corresponding to the coordinates of the centre of mass has the form

$$\frac{d}{dt} m \dot{\mathbf{R}}_C = - \frac{\partial U}{\partial \mathbf{R}_C} = - \nabla U = \mathbf{F}$$

whence we get the equation of motion of the centre of mass of the body:

$$m \ddot{\mathbf{R}}_C = \mathbf{F} \quad (26.3)$$

where \mathbf{F} is the resultant of the external forces acting on the body.

To obtain an equation determining the variation of the angular momentum \mathbf{M} of a body with time, let us remember that for an individual particle

$$\dot{\mathbf{M}}_\alpha = [\mathbf{r}_\alpha, \mathbf{F}_\alpha] = \mathbf{N}_\alpha$$

i.e. the time derivative of the angular momentum equals the moment of the force acting on the particle. Summation over all the particles of a body yields

$$\dot{\mathbf{M}} = \frac{d}{dt} \sum_{\alpha} \mathbf{M}_\alpha = \sum_{\alpha} [\mathbf{r}_\alpha, \mathbf{F}_\alpha] = \mathbf{N} \quad (26.4)$$

where \mathbf{N} is the sum of the moments of all the external forces acting on the body relative to the point C (the sum of the moments of the internal forces is zero).

Let us write Eq. (26.4) in projections onto the axes of the frame K' (onto the principal axes of inertia of a body):

$$\frac{d}{dt} M_i = N_i \quad (i = 1, 2, 3) \quad (26.5)$$

The projection N_i can be written as

$$N_i = - \frac{\delta U}{\delta \Phi_i} = - \frac{\partial U}{\partial \Phi_i} = \frac{\partial L}{\partial \Phi_i} \quad (26.6)$$

where $\delta \Phi_i$ is the angle of rotation of the body about its i -th principal axis ($\omega_i = d\Phi_i/dt$). Indeed, when the body rotates through the angle

$\delta\Phi_i$, the forces applied to it do the work

$$\delta A = \sum \mathbf{F} \delta \mathbf{R} = \sum \mathbf{F} [(\delta\Phi_i \mathbf{e}_i), \mathbf{r}]$$

Here summation is performed not over the subscript i , but over all the external forces acting on the body. $\delta \mathbf{R}$ is the displacement of the point of application of the relevant force. \mathbf{e}_i is the unit vector of the i -th principal axis, and \mathbf{r} is the position vector of the point of application of the force emerging from the point C (\mathbf{R} extends from the origin of the frame K). After a cyclic transposition of the factors and putting the common factor $\delta\Phi_i \mathbf{e}_i$ outside the sum sign, we arrive at the expression

$$\delta A = \delta\Phi_i \mathbf{e}_i \sum [\mathbf{r} \mathbf{F}] = \delta\Phi_i \mathbf{c}_i \mathbf{N} = N_i \delta\Phi_i$$

where N_i is the projection of the resultant moment of the forces onto the axis about which rotation through the angle $\delta\Phi_i$ has occurred. The work δA we have calculated is done at the expense of the decrement of the potential energy U , i.e.

$$\delta A = N_i \delta\Phi_i = -\delta U$$

whence follows formula (26.6).

We have obtained formula (26.6) in considering rotation about one of the principal axes of rotation of a body. This formula also holds in the most general case—for rotation about an arbitrary axis (naturally, provided that the force whose moment is being considered is a potential one).

A glance at formula (26.6) shows that the quantities N_i are generalized forces corresponding to the generalized coordinates Φ_i [see (4.20)].

Let us now differentiate function (26.1) with respect to ω_i . The result is

$$\frac{\partial L}{\partial \omega_i} = I_i \omega_i = M_i$$

[see formula (24.6)]. The projection of the angular velocity ω_i can be represented as $\dot{\Phi}_i$ [see the text following formula (26.6)]. We can therefore write that

$$M_i = \frac{\partial L}{\partial \omega_i} = \frac{\partial L}{\partial \dot{\Phi}_i} \quad (26.7)$$

[compare with formula (11.3)]. It follows from formula (26.7) that the quantities M_i are generalized momenta corresponding to the generalized coordinates Φ_i [see (4.19)].

Using relations (26.6) and (26.7), we can represent formula (26.5) as follows:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\Phi}_i} = \frac{\partial L}{\partial \Phi_i} \quad (26.8)$$

i.e. as Lagrange's equation corresponding to the generalized coordinate Φ_i .

It should be borne in mind that with our selection of the axes of the frame K' only the Euler angle ψ corresponds to rotation about a principal axis. The other angles correspond to rotations about the fixed axis X_3 and about the nodal line.

By differentiating function (26.2) with respect to $\dot{\psi}$, we shall find an expression for the projection of the angular momentum onto the axis x_3 (onto the z -axis) in terms of the Euler angles:

$$M_\psi = M_3 = I_3 (\dot{\varphi} \cos \vartheta + \dot{\psi}) \quad (26.9)$$

Assume that the Euler angle ϑ equals zero. This signifies that the Z - and z -axes coincide all the time—the body rotates about an axis associated with it that is fixed in the frame K . In this case, the sum of the angles $\varphi + \psi$ determines the total angle of rotation of the body about the z -axis. The position of the nodal line in this case is indefinite—it may be located at any place between the X - and x -axes. Particularly, the nodal line can be made to coincide with the X -axis. Now $\varphi = 0$, and rotation of the body about the z -axis will be characterized by the angle of proper rotation ψ . If we make the nodal line coincide with the x -axis, the angle ψ will vanish, and rotation about the z -axis will be described by the precession angle φ .

When $\vartheta = 0$, formula (26.9) becomes

$$M_3 = I_3 (\dot{\varphi} + \dot{\psi}) = I_3 \omega_3$$

where ω_3 is the angular velocity of rotation of the body about the z -axis.

When $\vartheta = \pi/2$, formula (26.9) is simplified as follows:

$$M_\psi = M_3 = I_3 \dot{\psi} = I_3 \omega_\psi$$

where ω_ψ is the angular velocity of proper rotation of the body.

In concluding, we shall compile Lagrange's equation for a body rotating about an axis rigidly associated with it that is fixed in the frame K . To be more general, we shall consider that the axis of rotation does not pass through the centre of mass C of the body and is not parallel to any of its principal axes of inertia.

We direct the axes Z and z along the axis of rotation of the body. Hence, the Euler angle ϑ will be zero. Since the nodal line is not fixed in this case, we shall make it coincide with the x -axis, and the angle ψ will also vanish. With our choice of the z -axis (along the direction of the vector ω), we have $\omega_x = \omega_y = 0$, and $\omega_z = \omega = \dot{\psi}$.

We have calculated the kinetic energy for such a case on an earlier page. It was equal to the value given by (23.20). Consequently, the

Lagrangian has the form

$$L = \frac{1}{2} I_{zz}^{(A)} \omega^2 - U(\varphi) = \frac{1}{2} I_{zz}^{(A)} \dot{\varphi}^2 - U(\varphi) \quad (26.10)$$

(A is a point not coinciding with C at which the origin of the frame K' is).

We compile Lagrange's equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial L}{\partial \varphi}$$

It follows from (26.10) that the left-hand side equals $I_{zz}^{(A)} \dot{\omega}$. By (26.6), the right-hand side, equal to $-\partial U/\partial \varphi$, is the projection of the resultant moment of the forces onto the axis of rotation, i.e. N_z . We thus arrive at the equation

$$I_{zz}^{(A)} \dot{\omega} = N_z$$

27. Euler's Equations

Lagrange's equations corresponding to the Euler angles (i.e. describing the rotation of a body), as can readily be concluded from the form of the function (26.2), are very complicated. It is sometimes more convenient to use other equations that were obtained by L. Euler and bear his name. To arrive at these equations, we shall proceed from relation (26.4):

$$\frac{d\mathbf{M}}{dt} = \mathbf{N} \quad (27.1)$$

Equation (27.1) holds in an inertial reference frame (i.e. in the stationary frame K). We shall try to find an equation that holds in the frame K' rotating together with the body whose axes coincide with the principal axes of inertia of the body.

In Eq. (27.1), $d\mathbf{M}$ is the increment of the vector \mathbf{M} during the time dt , observed in the frame K . By (15.6), this increment can be written as

$$d\mathbf{M} = d'\mathbf{M} + [d\varphi, \mathbf{M}]$$

where $d'\mathbf{M}$ is the increment of the vector \mathbf{M} during the time dt , observed in the frame K' , and $d\varphi$ is the angle through which the frame K' rotates during the time dt . Dividing the last equation by dt , we obtain

$$\frac{d\mathbf{M}}{dt} = \frac{d'\mathbf{M}}{dt} + \left[\frac{d\varphi}{dt}, \mathbf{M} \right] \quad (27.2)$$

where $d\mathbf{M}/dt$ is the rate of the change in the vector \mathbf{M} observed in the frame K , $d'\mathbf{M}/dt$ is the rate of change in the same vector observed in

the frame K' , and ω is the angular velocity of rotation of the frame K' (i.e. the angular velocity of rotation of the body).

Formula (27.2) holds for any vector, particularly, it also holds for the vector ω . In the latter case, we have

$$\frac{d\omega}{dt} = \frac{d'\omega}{dt} + [\omega\omega]$$

Since $[\omega\omega] = 0$, we arrive at the relation

$$\frac{d\omega}{dt} = \frac{d'\omega}{dt} \quad (27.3)$$

from which it follows that the rates of the change in the vector ω observed in the frames K and K' are identical.

Let us substitute expression (27.2) for the left-hand side of formula (27.1). The result is the equation

$$\frac{d'M}{dt} + [\omega M] = N$$

Let us project all the vectors onto the i -th axis of the frame K' , taking into account that the projection of the vector M onto this axis is $M_i = I_i \omega_i$ [see (24.6)]:

$$\frac{d'(I_i \omega_i)}{dt} + [\omega M]_{\text{pr. } x_i} = N_i$$

Knowing that $I_i = \text{const}$ and $d'\omega/dt = d\omega/dt$ [(see (27.3)], we can write the equation obtained as follows:

$$I_i \frac{d\omega_i}{dt} + [\omega M]_{\text{pr. } x_i} = N_i \quad (i = 1, 2, 3) \quad (27.4)$$

Let us represent the projection of the vector product onto the axis x_i by formula (VI.33):

$$[\omega M]_{\text{pr. } x_i} = \sum_{k, l} \varepsilon_{ikl} \omega_k M_l = \sum_{k, l} \varepsilon_{ikl} \omega_k I_l \omega_l$$

and introduce this expression into (27.4). As a result, we arrive at the equations

$$I_i \frac{d\omega_i}{dt} + \sum_{k, l} \varepsilon_{ikl} I_l \omega_k \omega_l = N_i \quad (i = 1, 2, 3) \quad (27.5)$$

which are Euler's equations. Assuming consecutively that $i = 1$, $i = 2$, $i = 3$, and summing over k and l , we obtain three equations:

$$\left. \begin{aligned} I_1 \frac{d\omega_1}{dt} + (I_3 - I_2) \omega_2 \omega_3 &= N_1 \\ I_2 \frac{d\omega_2}{dt} + (I_1 - I_3) \omega_3 \omega_1 &= N_2 \\ I_3 \frac{d\omega_3}{dt} + (I_2 - I_1) \omega_1 \omega_2 &= N_3 \end{aligned} \right\} \quad (27.6)$$

We must note that each following equation is obtained from the preceding one by means of a cyclic transposition of the subscripts 1, 2, 3.

It is easy to see that for a spherical top, Eqs. (27.6) transform into the equation $I\dot{\omega} = N$.

28. Free Symmetric Top

Let us use Euler's equations for studying the motion of a symmetric top (i.e. a body for which $I_1 = I_2 \neq I_3$) not experiencing the action of external forces. In this case, the centre of mass of the body moves at a constant velocity or is at rest [see formula (26.3)]. Therefore, from among all the inertial frames, we can choose the frame K whose origin coincides with the centre of mass C of the body. In this frame, translational motion of the body is absent, and it remains for us to establish only the nature of the rotation.

In the absence of external forces, the angular momentum M of the body remains constant in magnitude and direction relative to the stationary frame K . Let us choose the direction of the vector M as the Z -axis. We shall see that the vector M , generally speaking, constantly changes its direction relative to the frame K' associated with the body.

Since $I_1 = I_2$ and the moment of the external forces is zero, Eqs. (27.6) have the following form¹:

$$\left. \begin{aligned} I_1 \frac{d\omega_1}{dt} + (I_3 - I_1) \omega_2 \omega_3 &= 0 \\ I_1 \frac{d\omega_2}{dt} - (I_3 - I_1) \omega_3 \omega_1 &= 0 \\ \frac{d\omega_3}{dt} &= 0 \end{aligned} \right\} \quad (28.1)$$

We directly obtain from the third equation that $\omega_3 = \omega_{||} = \text{const.}$ This signifies that the projection of the angular velocity vector onto the z -axis associated with the body remains constant.

Introducing the symbol

$$\Omega = \frac{I_3 - I_1}{I_1} \omega_{||} \quad (28.2)$$

we can write the first two of Eqs. (28.1) as follows:

$$\dot{\omega}_1 = -\Omega \omega_2, \quad \dot{\omega}_2 = \Omega \omega_1$$

It is a simple matter to see that the system we have obtained is satisfied by the functions

$$\omega_1 = \omega_{\perp} \cos(\Omega t + \alpha), \quad \omega_2 = \omega_{\perp} \sin(\Omega t + \alpha) \quad (28.3)$$

¹ We remind our reader that Euler's equations are written in a coordinate system whose axes coincide with the principal axes of inertia of the body.

where ω_{\perp} and α are constants, $\omega_{\perp} = \sqrt{\omega_1^2 + \omega_2^2}$ being the magnitude of the projection of the vector ω onto the plane xy perpendicular to the axis of proper rotation of the body—the z -axis. Inspection of (28.3) shows that the component ω_{\perp} perpendicular to the z -axis, remaining constant in magnitude, rotates uniformly in the plane xy at the angular velocity Ω determined by formula (28.2). We have seen that the component ω_{\parallel} parallel to the z -axis also remains constant in magnitude. We thus conclude that the vector ω rotates relative to the body at the angular velocity Ω , describing a cone about the z -axis, the magnitude of the vector ω remaining unchanged (Fig. 28.1¹).

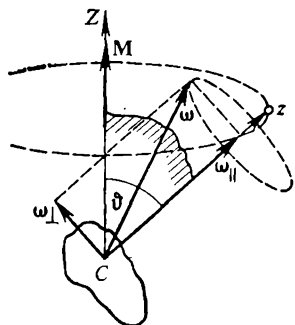


Fig. 28.1.

According to (24.6), the projections of the vector M onto the axes x , y , z are

$$M_x = I_1 \omega_1, \quad M_y = I_1 \omega_2, \quad M_z = I_3 \omega_3$$

Hence,

$$\begin{aligned} \mathbf{M} &= I_1 \omega_1 \mathbf{e}_x + I_1 \omega_2 \mathbf{e}_y + I_3 \omega_3 \mathbf{e}_z \\ &= I_1 (\omega_1 \mathbf{e}_x + \omega_2 \mathbf{e}_y) + I_3 \omega_3 \mathbf{e}_z \end{aligned}$$

where \mathbf{e}_x , \mathbf{e}_y , \mathbf{e}_z are the unit vectors of the relevant axes (these axes rotate together with the body). The sum $\omega_1 \mathbf{e}_x + \omega_2 \mathbf{e}_y$ gives the component ω_{\perp} perpendicular to the z -axis; $\omega_3 \mathbf{e}_z$ is the component ω_{\parallel} parallel to the z -axis.

Hence,

$$\mathbf{M} = I_1 \omega_{\perp} + I_3 \omega_{\parallel} \quad (28.4)$$

The directions of the vectors \mathbf{M} and ω_{\parallel} pass through a common origin for the frames K and K' , i.e. the point C . Consequently, these vectors determine a certain plane (the plane Zz). For equality (28.4) to be observed, the vector ω_{\perp} must be in the same plane. Hence, the vector $\omega = \omega_{\perp} + \omega_{\parallel}$ is in the plane Zz . We thus conclude that the vectors \mathbf{M} , ω , and the axis z of proper rotation of the body are in the same plane at each instant (this plane is hatched in Fig. 28.1). The plane rotates about the direction of \mathbf{M} . The z -axis rotates together with it and describes about the Z -axis a circular cone. Such rotation of the body's z -axis is called **regular precession**. The latter is characterized by a constant nutation angle ϑ .

We have established that the vector ω rotates relative to the body about the z -axis at the angular velocity Ω . At the same time, this vector remains in the plane Zz . It thus follows that relative to

¹ In practice, a body of revolution is taken as a symmetric top. We have depicted a body of an irregular shape in the figure, however, to underline the fact that the only condition for a body to be a symmetric top is the equality of two of its principal moments of inertia.

the plane Zz , the body rotates about the z -axis in the opposite direction at the same velocity Ω .

Figure 28.2 shows the components $I_1\omega_\perp$ and $I_3\omega_\parallel$ whose sum gives the angular momentum M [see (28.4)]. A glance at the figure shows that

$$\tan \vartheta = \frac{I_1\omega_\perp}{I_3\omega_\parallel} \quad (28.5)$$

We established earlier that ω_\perp and ω_\parallel are constants. Therefore, $\tan \vartheta$, and, consequently, the nutation angle ϑ itself remain constant. The angle between the vector ω and the z -axis is also constant (its tangent is $\omega_\perp/\omega_\parallel$).

The quantities ω_\perp and ω_\parallel are determined by the body's kinetic energy T and the angular momentum M . Indeed,

$$\left. \begin{aligned} T &= \frac{1}{2} (I_1\omega_1^2 + I_1\omega_2^2 + I_3\omega_3^2) = \frac{1}{2} (I_1\omega_\perp^2 + I_3\omega_\parallel^2) \\ M^2 &= (I_1\omega_1)^2 + (I_1\omega_2)^2 + (I_3\omega_3)^2 = I_1^2\omega_\perp^2 + I_3^2\omega_\parallel^2 \end{aligned} \right\} \quad (28.6)$$

Solving this system of equations for ω_\perp and ω_\parallel , we find the expressions of these quantities in terms of T and M :

$$\omega_\parallel = \sqrt{\frac{M^2 - 2TI_1}{I_3(I_3 - I_1)}}, \quad \omega_\perp = \sqrt{\frac{M^2 - 2TI_3}{I_1(I_1 - I_3)}} \quad (28.7)$$

Assume that the body is flattened along the z -axis. Hence, $I_3 > I_1$, and the denominator in the radicand of the expression for ω_\parallel will be positive and for ω_\perp , negative. Consequently, for ω_\parallel and ω_\perp to be real, it is necessary that the conditions $M^2 - 2TI_1 \geq 0$ and $M^2 - 2TI_3 \leq 0$ be satisfied; they can be combined into a single formula:

$$\frac{M^2}{2I_3} \leq T \leq \frac{M^2}{2I_1} \quad (28.8)$$

The energy of a free symmetric top cannot have values beyond the indicated limits (at a given M). If the energy has its lowest possible value, i.e. at $M^2 = 2TI_3$, the quantity ω_\perp vanishes [see (28.7)]. Examination of formula (28.5) shows that in this case $\vartheta = 0$ —the axes Z and z coincide, the directions of the vectors M and ω also coincide, and the vector ω does not move relative to the body or relative to the frame K . We must note that in this case the following relation holds:

$$T = \frac{M^2}{2I_3} \quad (28.9)$$

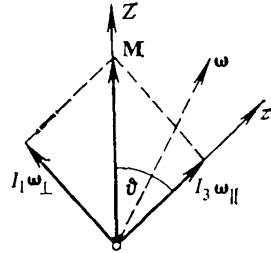


Fig. 28.2.

axis of inertia of the body. Hence, the axes ω_i coincide with the axes x, y, z . Rotation relative to the axes ω_i thus signifies rotation relative to the body itself.

We can increase M with T remaining constant until the energy and angular momentum ellipsoids have only two common points. This occurs provided that $\sqrt{2T/I_3} = M/I_3$ [compare with (28.9)]. In this case, the vector ω coincides in direction with the z -axis. If we continue to increase M , i.e. assume that $M^2/2I_3 > T$, the ellipsoids stop touching each other, and the system of equations (28.11) and (28.12) has no common solutions. Such a case cannot be realized. Consequently, we have obtained the lower boundary for T [see (28.8)].

By diminishing M with T remaining constant, we arrive at a situation when both lines of intersection of the ellipsoids merge into a single line of contact in the equatorial plane. This occurs provided that $\sqrt{2T/I_1} = M/I_1$ [compare with (28.10)]. In this case, the vector ω rotates about the z -axis, constantly remaining perpendicular to it. If we continue to decrease M , i.e. assume that $T > M^2/2I_1$, the ellipsoids stop touching each other, and the system of equations (28.11) and (28.12) has no common solutions. We have thus arrived at the upper boundary for T [see (28.8)].

29. Symmetric Top in a Homogeneous Gravitational Field

Consider the behaviour of a symmetric top with one fixed point in a homogeneous gravitational field. We must note that a general solution¹ of the problem on the motion of a body with one fixed point in a homogeneous gravitational field can be obtained only in three cases:

(1) for an asymmetric balanced top (a top is called balanced if the fixed point coincides with the centre of mass of the top). This case is known as Euler's problem;

(2) for a symmetric unbalanced top (the fixed point does not coincide with the centre of mass) in which the centre of mass is on the z -axis—Lagrange's problem;

(3) for a symmetric unbalanced top for which $I_1 = I_2 = 2I_3$, and the centre of mass is in the plane xy —the problem of S. Kovalevskaya.

We shall consider Lagrange's problem. The equations of motion in this case are integrated in a very complicated way. We shall therefore limit ourselves to writing the initial equations and discussing their solutions.

We place the origins of both coordinate systems K and K' at the fixed point A of the top (at the point of support). We direct the

¹ That is, the solution obtained with the aid of quadratures with arbitrary initial conditions.

Z-axis of the stationary frame K along a vertical line, and the z -axis of the frame K' associated with the top along its third principal axis of inertia ($I_1 = I_2 \neq I_3$). With such a choice of the coordinate axes, the top's potential energy has the form $U = mgl \cos \vartheta$, where l is the distance from the supporting point to the centre of mass C (we assume that the coordinate z of the centre of mass, i.e. z_C , is greater than zero).

Let us find the expression for the kinetic energy in the given case. With a view to the point A being fixed, we can write

$$T = \frac{1}{2} \sum_{\alpha} m_{\alpha} [\omega, \mathbf{r}_{\alpha}^{(A)}]^2$$

and perform the same transformations that we made for expression (23.3). As a result, we arrive at a formula which will differ from (23.5) only in containing the coordinates $x_{i\alpha}^{(A)}$, $x_{h\alpha}^{(A)}$, etc. instead of the coordinates $x_{i\alpha}$, $x_{h\alpha}$, etc. Consequently, we obtain an expression similar to (23.8) for the kinetic energy:

$$T = \frac{1}{2} \sum_{i, h} I_{ih}^{(A)} \omega_i \omega_h$$

where $I_{ih}^{(A)}$ are the tensor components determined by formula (23.16). According to (23.18), these components are

$$I_{ih}^{(A)} = I_{ih} + m(a^2 \delta_{ih} - a_i a_h)$$

In our case, $a_1 = a_2 = 0$, and $a_3 = -a = -l$ [here a_i is the i -th coordinate of the point A in the frame $K'_{(C)}$, see (23.17)]. In addition, since the z -axis coincides with the third principal axis of inertia, while the axes x and y are parallel to the two other principal axes¹, we have $I_{ih} = I_i \delta_{ih}$. We can thus conclude that the tensor $I_{ih}^{(A)}$ is diagonal, its non-zero components being

$$I_1^{(A)} = I_1 + ml^2, \quad I_2^{(A)} = I_2 + ml^2, \quad I_3^{(A)} = I_3$$

Hence, taking into account that $I_1 = I_2$, we obtain the following expression for the kinetic energy:

$$T = \frac{1}{2} [(I_1 + ml^2) (\omega_1^2 + \omega_2^2) + I_3 \omega_3^2]$$

Introducing values (22.4) for the components of ω into this expression, we obtain

$$T = \frac{1}{2} [(I_1 + ml^2) (\dot{\varphi}^2 \sin^2 \vartheta + \dot{\vartheta}^2) + I_3 (\dot{\varphi} \cos \vartheta + \dot{\psi})^2]$$

¹ In a symmetric top, any two mutually perpendicular axes perpendicular to the axis of symmetry may be principal axes of inertia.

Let us write the Lagrangian¹

$$L = \frac{1}{2} (I_1 + ml^2) (\dot{\varphi}^2 \sin^2 \vartheta + \dot{\vartheta}^2) + \frac{1}{2} I_3 (\dot{\varphi} \cos \vartheta + \dot{\psi})^2 - mgl \cos \vartheta \quad (29.1)$$

The coordinates φ and ψ are cyclic [see the text in Sec. 11 related to formula (11.2)]. Therefore, the generalized momenta p_φ and p_ψ

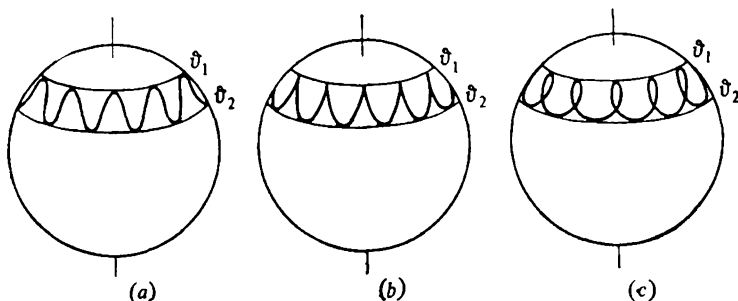


Fig. 29.1.

are integrals of motion. The energy E is a third integral of motion. We thus have three equations:

$$p_\varphi = \partial L / \partial \dot{\varphi} = [(I_1 + ml^2) \sin^2 \vartheta + I_3 \cos^2 \vartheta] \dot{\varphi} + I_3 \cos \vartheta \dot{\psi} = M_z = \text{const}$$

$$p_\psi = \partial L / \partial \dot{\psi} = I_3 (\dot{\varphi} \cos \vartheta + \dot{\psi}) = M_z = \text{const}^2$$

$$E = T + U = \text{const}$$

An analysis of the solutions of these equations leads to the following results. The angle ϑ varies periodically within the limits from ϑ_1 to ϑ_2 determined by the initial conditions (particularly, by the relation between the energy and the angular momentum of the top). The oscillations of the top's axis corresponding to the variations of the angle ϑ are called **nutation**. Simultaneously, the axis of the top precesses, i.e. turns about the Z -axis. As a result, the apex, i.e. the point of intersection of the z -axis (the axis of the top) and a sphere of unit radius, describes one of the curves shown in Fig. 29.1. The sign of the derivative $\dot{\varphi}$ either remains unchanged (Fig. 29.1a and b) or changes (Fig. 29.1c). Case (b) occurs when $\dot{\varphi}$ and $\dot{\vartheta}$ simultaneously vanish.

¹ The force applied to a top at its point of support is a reaction of the constraint, which, as we know, is not included in Lagrange's equations.

² Compare with (26.9).

The nature of the behaviour of $\dot{\phi}$, like the values of θ_1 and θ_2 , depends on the initial conditions. Motion of the top's axis such as that in case (b) corresponds to natural initial conditions when the top is first brought into rotation about its axis, after which the axis is released and begins its motion. At the moment of axis release, both $\dot{\phi}$ and $\dot{\theta}$ equal zero. In these initial conditions, the top first inclines, and then upon reaching the boundary angle θ_2 it begins to rise (see Fig. 29.1b).

In absolutely specific initial conditions, both boundary values of θ_1 and θ_2 coincide so that the top's axis precesses without nutation. As we have already noted, such precession is called regular. To obtain regular precession, a top must be given an initial impetus of a quite definite magnitude and direction.

For a "rapid" top (i.e. a top whose kinetic energy of proper rotation is high compared with its energy in a gravitational field), the action of gravitational forces may be disregarded in a first approximation. Consequently, the motion of the top can be represented as the free precession of the top's axis about the direction of the angular momentum \mathbf{M} , considered in Sec. 28 (this precession corresponds to nutation of a heavy top), onto which small disturbances due to the action of the gravitational force are superposed. These disturbances cause slow precession of the angular momentum \mathbf{M} about a vertical line.

Calculations show that the more rapidly a top rotates, the smaller is the amplitude of nutation. In addition, in a real rapid top, nutation is damped by friction at the support. Therefore, in practice the nutation of a sufficiently rapid top is sometimes unnoticeable, and the top seems to uniformly precess about a vertical axis. Since such precession is regular only approximately, it is known as pseudo-regular precession.

30. Hamilton's Equations

In solving problems on the motion of a system with s degrees of freedom using Lagrange's equations, we have to solve a system of s second-order differential equations. The generalized coordinates q_k and the generalized velocities \dot{q}_k are the independent variables in these equations.

W. Hamilton obtained equations of motion in which the generalized coordinates q_k and the generalized momenta p_k are the independent variables. Hamilton's equations or, as they are also called, canonical¹ equations (q_k and p_k are accordingly called canonical variables), unlike Lagrange's equations, are first-order differential ones. But on the other hand, their number needed to describe a system with s degrees of freedom is $2s$.

Hamilton's equations can be derived either from Lagrange's equations, or directly from the principle of least action (we shall give both derivations below). It is natural that they give nothing novel in essence. But canonical equations are more symmetric than Lagrange's equations and, in addition, being invariant with respect to canonical transformations, they unveil great possibilities for generalizations playing an important role in electrodynamics, statistical physics, and quantum mechanics.

Hamilton took the energy (5.1) expressed in terms of the variables q_k and p_k as the function characterizing a mechanical system. Having in view that

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \quad (30.1)$$

¹ Hamilton's equations are called canonical because they remain invariant upon quite general transformations of the variables. With the aid of such canonical transformations, we can pass over from the variables q_k and p_k to other canonical variables: $Q_i(q_k, p_k, t)$ and $P_i(q_k, p_k, t)$. Here Hamilton's equations retain their form, true, with the new Hamiltonian $H'(Q_i, P_i, t)$ that replaces the function $H(q_k, p_k, t)$. The variables Q_i and P_i may have a different physical meaning than the variables q_k and p_k .

[see (4.19)], let us write this function as follows:

$$H(q_k, p_k, t) = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t) \quad (30.2)$$

(\dot{q}_k is assumed to be expressed in terms of q_k and p_k). The characteristic function H is called the **Hamiltonian function** or simply the **Hamiltonian**.

We shall give as an example the Hamiltonian of a particle moving in the potential field $U = U(x, y, z, t)$:

$$H = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + U(x, y, z, t) \quad (30.3)$$

For a particle moving in a stationary field, H has the same form, but U does not depend explicitly on t .

Let us derive Hamilton's equations proceeding from those of Lagrange. For this purpose, we shall find the total differentials of the left-hand and right-hand sides of formula (30.2) and equate them to each other. The total differential of the left-hand side is

$$dH = \sum_k \frac{\partial H}{\partial q_k} dq_k + \sum_k \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt \quad (30.4)$$

The differential of the right-hand side is

$$dH = \sum_k p_k d\dot{q}_k + \sum_k \dot{q}_k dp_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt \quad (30.5)$$

In view of relation (30.1), the first and fourth sums cancel each other. It follows from Lagrange's equation (4.16)¹ that

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \dot{p}_k \quad (30.6)$$

Let us substitute \dot{p}_k for $\partial L / \partial q_k$ in the third sum of formula (30.5). As a result, expression (30.5) becomes

$$dH = \sum_k \dot{q}_k dp_k - \sum_k \dot{p}_k dq_k - \frac{\partial L}{\partial t} dt \quad (30.7)$$

For expressions (30.4) and (30.7) to be equal at arbitrary values of dq_k , dp_k , and dt , the following conditions must be ob-

¹ We assume that all the forces acting in the system are potential ones; this is why we use Eq. (4.16) instead of (4.15).

served:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} \quad (k = 1, 2, \dots, s) \quad (30.8)$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad (30.9)$$

Equations (30.8) are the required Hamilton or canonical equations. As we have already noted, they are first-order differential equations. Their total number is $2s$.

For a particle described by the Hamiltonian (30.3), Hamilton's equations have the form

$$\dot{p}_x = -\frac{\partial U}{\partial x}, \quad \dot{x} = \frac{p_x}{m}, \text{ etc.}$$

Now let us derive Hamilton's equations from the principle of least action. We remind our reader that according to this principle, a system moves so that the action S [see (7.1)] has the smallest possible value. This statement is written in the form

$$\delta S = \delta \int_{t_1}^{t_2} L(q_k, \dot{q}_k, t) dt = 0$$

Let us introduce into this equation the value of L obtained from relation (30.2):

$$\delta \int_{t_1}^{t_2} \left(\sum_k p_k \dot{q}_k - H \right) dt = 0 \quad (30.10)$$

The variation on the left-hand side of (30.10) can be written as

$$\delta S = \int_{t_1}^{t_2} \sum_k \left(p_k \delta \dot{q}_k + \dot{q}_k \delta p_k - \frac{\partial H}{\partial q_k} \delta q_k - \frac{\partial H}{\partial p_k} \delta p_k \right) dt$$

Let us integrate the first term by parts:

$$\int_{t_1}^{t_2} p_k \delta \dot{q}_k dt = p_k \delta q_k \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_k \delta q_k dt$$

Here we have taken advantage of the fact that $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$ [see (III.4)]. The variations δq_k vanish when the integration limits

are introduced¹. Therefore, the first term of the expression obtained must be discarded. Consequently, condition (30.10) will be written as follows:

$$\int_{t_1}^{t_2} \sum_k \left\{ \left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right) \delta q_k \right\} dt = 0$$

Owing to the arbitrary nature of the variations δq_k and δp_k , this condition can be satisfied only if the expressions in parentheses will be zeros. Hence we directly obtain Eqs. (30.8).

Let us investigate the Hamiltonian H . We find the total derivative of this function with respect to time:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_k \frac{\partial H}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial H}{\partial p_k} \dot{p}_k$$

Taking into account values (30.8) for \dot{q}_k and \dot{p}_k , we find that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \quad (30.11)$$

Thus, if the function H does not depend explicitly on the time, it retains its value. This was to be expected because H is the total energy of a system that is retained provided that

$$\partial L / \partial t = 0 \quad [\text{see (30.9)}].$$

Let us replace \dot{p}_k in the second of equations (30.8) in accordance with (30.6). As a result, we find that

$$\frac{\partial H}{\partial q_k} = - \frac{\partial L}{\partial q_k} \quad (30.12)$$

It thus follows that the generalized coordinates which are cyclic, i.e. which are not contained explicitly in the function L , are also not contained explicitly in the function H . On an earlier page [see (11.2)], we established that the generalized momenta corresponding to cyclic coordinates are integrals of motion. We can conclude from what has been said that the generalized momenta corresponding to the coordinates q_k not included explicitly in the Hamiltonian (i.e. cyclic relative to the function H) remain constant:

$$p_k = \text{const provided that } \frac{\partial H}{\partial q_k} = 0 \quad (30.13)$$

¹ We remind our reader that in the variation of trajectories in a configuration space (see Sec. 4), the initial and final points of the trajectories are assumed to be fixed.

31. Poisson Brackets

Let us take a function of the canonical variables q_k and p_k , and also of the time t , that is, $f(q_k, p_k, t)$ and establish in what conditions this function will be an integral of motion¹. To do this, we calculate the total derivative of this function with respect to time:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q_k} \dot{q}_k + \frac{\partial f}{\partial p_k} \dot{p}_k \right)$$

We substitute the values for the derivatives \dot{q}_k and \dot{p}_k from (30.8):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \quad (31.1)$$

If we have two functions $\varphi(q_k, p_k, t)$ and $\psi(q_k, p_k, t)$, the expression

$$\{\varphi, \psi\} = \sum_k \left(\frac{\partial \varphi}{\partial q_k} \frac{\partial \psi}{\partial p_k} - \frac{\partial \varphi}{\partial p_k} \frac{\partial \psi}{\partial q_k} \right) \quad (31.2)$$

is called the **Poisson bracket** for the functions φ and ψ ². When necessary, the independent variables with respect to which the partial derivatives are taken are indicated as subscripts of the symbol of the Poisson bracket, i.e., for instance, bracket (31.2) is written as $\{\varphi, \psi\}_{q,p}$. It is easy to see that

$$\{\varphi, \psi\}_{q,p} = -\{\varphi, \psi\}_{p,q} = \{\psi, \varphi\}_{p,q} \quad (31.3)$$

Using the Poisson bracket, expression (31.1) can be represented as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}_{q,p} \quad (31.4)$$

or as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\}_{p,q} \quad (31.5)$$

A glance at (31.4) shows that the condition in which the function f

¹ Do not forget that the functions of dynamic variables (q_k and \dot{q}_k or q_k and p_k) that remain constant in the motion of a system are known as integrals of motion.

² Sometimes square brackets and even round ones (parentheses) are used to designate the Poisson bracket instead of curly brackets (braces).

is an integral of motion is

$$\frac{\partial f}{\partial t} + \{f, H\} = 0 \quad (31.6)$$

We thus conclude that when the integral of motion f does not depend explicitly on the time, its Poisson bracket with the Hamiltonian equals zero.

Below are given some obvious properties of the Poisson bracket:

$$\{\varphi, \psi\} = -\{\psi, \varphi\} \quad (31.7)$$

$$\{\varphi, \varphi\} = 0 \quad (31.8)$$

$$\{(\varphi_1 + \varphi_2), \psi\} = \{\varphi_1, \psi\} + \{\varphi_2, \psi\} \quad (31.9)$$

$$\{(\varphi_1 \varphi_2), \psi\} = \varphi_1 \{\varphi_2, \psi\} + \varphi_2 \{\varphi_1, \psi\} \quad (31.10)$$

$$\frac{\partial}{\partial t} \{\varphi, \psi\} = \left\{ \frac{\partial \varphi}{\partial t}, \psi \right\} + \left\{ \varphi, \frac{\partial \psi}{\partial t} \right\} \quad (31.11)$$

Particularly, we may take canonical variables as φ or ψ , or as both of them. This yields the following relations:

$$\{q_i, \psi\} = \frac{\partial \psi}{\partial p_i} \quad (31.12)$$

$$\{p_i, \psi\} = -\frac{\partial \psi}{\partial q_i} \quad (31.13)$$

$$\{q_i, q_k\} = 0 \quad (31.14)$$

$$\{p_i, p_k\} = 0 \quad (31.15)$$

$$\{q_i, p_k\} = \delta_{ik} \quad (31.16)$$

We invite our reader to obtain formulas (31.12)-(31.16) as an exercise. In deriving them, take into account that $\partial p_i / \partial q_k = 0$, and $\partial q_i / \partial p_k = 0$ (because q_k and p_k are independent variables).

A very important property of the Poisson bracket is its invariance relative to canonical transformations. This signifies that

$$\{\varphi, \psi\}_{q, p} = \{\varphi, \psi\}_{Q, P} \quad (31.17)$$

where Q and P are variables obtained from q and p with the aid of canonical transformations.

In quantum mechanics, we shall acquaint ourselves with the quantum Poisson bracket that is the quantum mechanical analogue of the classical Poisson bracket treated in this section.

32. The Hamilton-Jacobi Equation

Variation of the action

$$S = \int_{t_1}^{t_2} L dt \quad (32.1)$$

in finding the true trajectory of motion of a system (we have in mind the trajectory in configuration space, i.e. in a space with s dimensions; s is the number of degrees of freedom) consists in comparing the values of S for close trajectories with fixed ends, i.e. with

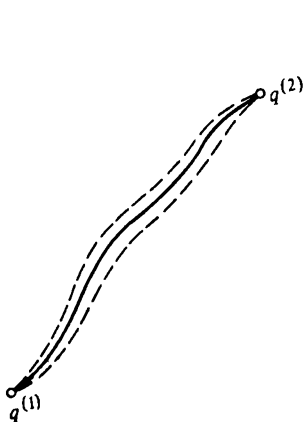


Fig. 32.1.

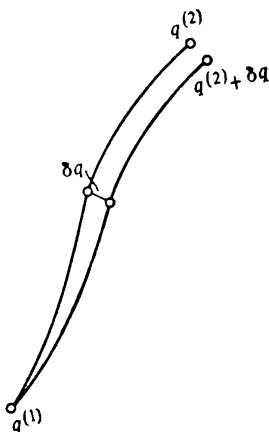


Fig. 32.2.

identical values of $q_k(t_1) = q_k^{(1)}$ and $q_k(t_2) = q_k^{(2)}$. This can be illustrated with the aid of Fig. 32.1. Only the trajectory for which S is minimum corresponds to actual motion (it is depicted by a solid line in the figure).

In the present section, we shall consider the action S as a quantity characterizing motion along true trajectories, and study the behaviour of this quantity upon changes in the point $q^{(2)}$ (with $t_2 = \text{const}$), and also upon changes in t_2 [the symbol $q^{(2)}$ signifies the set of all the $q_k^{(2)}$'s]. We shall thus treat the action as the function

$$S = S(q_k, t) \quad (32.2)$$

where q_k are the coordinates of the final position of the system, and t is the instant when this position is reached.

Let us take near the point $q^{(2)}$ a point with the coordinate $q^{(2)} + \delta q$ which the system reaches at the same instant t_2 in which it

arrives at the point $q^{(2)}$ (Fig. 32.2). The action for the trajectory bringing the system to the point $q^{(2)} + \delta q$ differs from the action for the trajectory bringing the system to the point $q^{(2)}$ by the quantity

$$\delta S = \int_{t_1}^{t_2} \sum_k \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) dt \quad (32.3)$$

Here δq_k is the difference between the values of q_k taken for both trajectories at the same instant t ; similarly, $\delta \dot{q}_k$ is the difference between the values of \dot{q}_k at the instant t .

Let us integrate the second term in (32.3) by parts:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k dt = \frac{\partial L}{\partial \dot{q}_k} \delta q_k \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k dt \quad (32.4)$$

For the true trajectory, $\partial L / \partial \dot{q}_k$ is the generalized momentum p_k . The origins of both trajectories coincide, hence $\delta q_k(t_1) = 0$. The quantity $\delta q_k(t_2)$ can be designated simply by δq_k . Consequently, the first term in (32.4) can be represented in the form $p_k \delta q_k$.

Let us introduce (32.4) into expression (32.3):

$$\delta S = \sum_k p_k \delta q_k + \int_{t_1}^{t_2} \sum_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k dt$$

True trajectories satisfy Lagrange's equations. Therefore the integrand and, consequently, the integral itself, vanishes. We have thus obtained the following value for the increment of the action S due to the change in the coordinates of the final position of the system by δq_k (at a constant time of motion):

$$\delta S = \sum_k p_k \delta q_k \quad (32.5)$$

Here p_k is the value of the momentum at the instant t_2 .

It follows from expression (32.5) that

$$\frac{\partial S}{\partial q_k} = p_k \quad (32.6)$$

Consequently, the partial derivatives of the action with respect to the generalized coordinates equal the corresponding generalized momenta.

Now assume that the upper limit of integration in (32.1) is not fixed. To underline this fact, we shall write the action in the form

$$S = \int_{t_1}^t L dt \quad (32.7)$$

The action represented in this way is a function of the upper integration limit, i.e. $S = S(t)$. It can be seen from (32.7) that

$$\frac{dS}{dt} = L \quad (32.8)$$

At the same time, in accordance with (32.2), we can write that

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_k \frac{\partial S}{\partial q_k} \dot{q}_k = \frac{\partial S}{\partial t} + \sum_k p_k \dot{q}_k \quad (32.9)$$

[we have taken relation (32.6) into account]. Equating the right-hand sides of expressions (32.8) and (32.9), we get the following value for the partial derivative of S with respect to t :

$$\frac{\partial S}{\partial t} = - \left(\sum_k p_k \dot{q}_k - L \right)$$

The expression in parentheses is the Hamiltonian H . Hence,

$$\frac{\partial S}{\partial t} = -H(q_k, p_k, t) \quad (32.10)$$

In accordance with formulas (32.6) and (32.10), the differential of function (32.2) can be written as

$$dS = \sum_k p_k dq_k - H dt \quad (32.11)$$

Let us substitute for the p_k 's in Eq. (32.10) their values from (32.6) and write this equation as follows:

$$\frac{\partial S}{\partial t} + H \left(q_1, q_2, \dots, q_s; \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \dots, \frac{\partial S}{\partial q_s}; t \right) = 0 \quad (32.12)$$

We have obtained a differential equation that must be satisfied by the function $S(q_1, q_2, \dots, q_s; t)$. It is called the **Hamilton-Jacobi equation**. It is an equation in partial derivatives of the first order.

Equation (32.12) is the cornerstone of a general method of integrating equations of motion. But a treatment of this method is beyond the scope of our course.

For a conservative system with stationary constraints, the time is not contained explicitly in the function H , and $H = E = \text{const}$ [see (30.9)]. Consequently, according to (32.10), the dependence of S on t is expressed by the term $-Et$. Therefore, the action breaks up into two terms, one of which depends only on the generalized coordinates, and the other only on the time:

$$S(q_k, t) = S_0(q_k) - Et \quad (32.13)$$

The function $S_0(q_k)$ is called the **contracted action**. Introducing S in the form of (32.13) into Eq. (32.12), we arrive at the Hamilton-

Jacobi equation for contracted action:

$$H\left(q_1, q_2, \dots, q_s; \frac{\partial S_0}{\partial q_1}, \frac{\partial S_0}{\partial q_2}, \dots, \frac{\partial S_0}{\partial q_s}\right) = E \quad (32.14)$$

The Hamilton-Jacobi equation plays an important role in optics and quantum mechanics. It underlies optical-mechanical analogy, which led E. Schrödinger to the formulation of wave mechanics.

Let us write the Hamilton-Jacobi equation for a particle moving in a non-stationary potential field. Taking into account formulas (30.3) and (32.12), we obtain

$$\frac{1}{2m} \left\{ \left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 \right\} + U(x, y, z, t) = -\frac{\partial S}{\partial t} \quad (32.15)$$

If the field in which the particle is moving is stationary, instead of (32.15), an equation for the contracted action S_0 is considered:

$$\frac{1}{2m} \left\{ \left(\frac{\partial S_0}{\partial x} \right)^2 + \left(\frac{\partial S_0}{\partial y} \right)^2 + \left(\frac{\partial S_0}{\partial z} \right)^2 \right\} + U(x, y, z) = E \quad (32.16)$$

Chapter VII

THE SPECIAL THEORY OF RELATIVITY

33. The Principle of Relativity

The special theory of relativity is based on two postulates formulated by Albert Einstein:

1. *All laws of nature are the same in all inertia reference frames.* In other words, we can say that *the equations expressing the laws of nature are invariant¹ with respect to transformations of coordinates and time from one inertial reference frame to another.*

2. *Light always propagates in a vacuum at a definite constant speed c not depending on the state of motion of the emitting body.*

The first postulate is called **Einstein's principle of relativity**, and the second is called the **principle of constancy of the speed of light**.

Newtonian mechanics proceeds from the assumption that interactions are transmitted instantaneously from body to body. This, particularly, manifests itself in that the interaction of particles is described with the aid of the potential energy $U(\mathbf{r}_1, \mathbf{r}_2, \dots)$, which depends only on the coordinates of the particles. It is thus assumed that a change in the position of one of the particles affects the other particles at the very same instant.

Actually, as shown by experiments, there are no instantaneous interactions in nature. If the position of a particle changes, this change begins to tell on another particle interacting with it after a finite interval of time elapses, needed for the interaction propagating at a finite velocity to cover a path equal to the distance between the particles. Consequently, we must acknowledge the existence of a maximum velocity of propagation of interactions. Experiments show that this velocity equals c —the speed of light in a vacuum.

It also follows from the second postulate that the velocity of propagation of interactions is the same in all inertial reference frames, i.e. is a universal constant.

In accordance with Galileo's mechanical principle of relativity, the laws of mechanics are invariant relative to the Galilean transformations:

$$x = x' + v_0 t', \quad y = y', \quad z = z', \quad t = t' \quad (33.1)$$

¹ The invariance of an equation signifies that the form of the equation does not change when the coordinates and time of one reference frame are replaced in it with the coordinates and time of another frame.

(v_0 is the velocity of the frame K' relative to the frame K). These transformations lead to the classical law of velocity summation:

$$\mathbf{v} = \mathbf{v}' + \mathbf{v}_0 \quad (33.2)$$

The latter equation and, consequently, Eqs. (33.1) from which it follows, does not agree with Einstein's second postulate according to which $c = c'$ for a light signal.

Consider two inertial reference frames K and K' . We shall select the coordinate axes of these frames so that the axes x and x' are directed along the velocity \mathbf{v}_0 of the frame K' , and the axes y and z

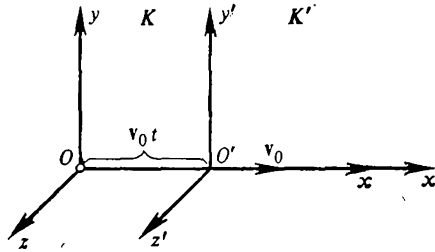


Fig. 33.1.

are parallel to the axes y' and z' (Fig. 33.1). We shall begin to measure the time in both frames from the instant when the origins of the systems coincide. Assume that a light signal propagating in all directions was sent at the instant $t = t' = 0$ from the coinciding origins of coordinates. By the instant t , the signal in the frame K will reach points at the distance $l = ct$ from O . The coordinates of these points satisfy the equation

$$c^2 t^2 - x^2 - y^2 - z^2 = 0 \quad (33.3)$$

Similarly, by the instant t' , the signal in the frame K' will reach points of a sphere of the radius ct' . The coordinates of these points satisfy the equation

$$c^2 t'^2 - x'^2 - y'^2 - z'^2 = 0 \quad (33.4)$$

Equations (33.3) and (33.4) have the same form, which manifests the invariance of the law of propagation of light with respect to a transformation of coordinates and time from one frame to another. If we introduce into (33.3) the values of the unprimed coordinates and time determined by formulas (33.1), we get the relation

$$c^2 t'^2 - x'^2 - y'^2 - z'^2 - 2v_0 x' t' - v_0^2 t'^2 = 0$$

not coinciding with (33.4). Hence, we have again arrived at the conclusion that the Galilean transformations are not compatible with the principle of the constancy of the speed of light.

According to Einstein's principle of relativity, all the laws of nature, including the laws of mechanics and electrodynamics, must be invariant relative to the same transformations of the coordinates and time performed in passing from one reference frame to another. But Newton's equations and Maxwell's equations do not meet this requirement. Whereas Newton's equations are invariant with respect to the Galilean transformations, Maxwell's equations, as can readily be seen by direct verification, are not invariant relative to these transformations. This circumstance led Einstein to the conclusion that Newton's equations need refinement, as a result of which the laws of mechanics and electrodynamics would be invariant with respect to the same transformations. The required modification of the laws of mechanics was performed by Einstein. The result was the appearance of mechanics agreeing with Einstein's principle of relativity, which was given the name of **relativistic mechanics**. The present chapter is devoted to a treatment of the fundamentals of this mechanics.

34. Interval

An event occurring with a particle is characterized by the place where it occurred (i.e. a set of values of x, y, z) and the time t when it occurred. If we introduce an imaginary four-dimensional space along whose axes we lay off the space coordinates x, y, z and the time t (or a quantity proportional to t), an event will be characterized in this space by a point. A point depicting an event in a four-dimensional space is called a **world point**. With time, a world point corresponding to a given particle moves in four-dimensional space, describing a line known as a **world line**.

Let us consider two events, the first of which consists in the emission of a light signal from a point with the coordinates x_1, y_1, z_1 at the instant t_1 , and the second in the arrival of this signal at a point with the coordinates x_2, y_2, z_2 at the instant t_2 . The following relation holds between the coordinates and time of these two events:

$$c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0 \quad (34.1)$$

The quantity

$$l_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad (34.2)$$

is the square of the distance (or of the interval) between two points in conventional space. We can speak in a similar way about the distance (interval) between two points in a four-dimensional space. The interval between two events is defined to be the quantity s_{12} whose square is determined by the formula

$$\begin{aligned} s_{12}^2 &= c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 \\ &= c^2 (t_2 - t_1)^2 - l_{12}^2 \end{aligned} \quad (34.3)$$

For two infinitely close events, the square of the interval is

$$ds^2 = c^2 dt^2 - dl^2 \quad (34.4)$$

For two events consisting in the emission of a light signal at one point and its arrival at another point, the interval is zero:

$$\Delta s^2 = c^2 \Delta t^2 - \Delta l^2 = 0 \quad (34.5)$$

[see formula (34.1)]. Owing to the constancy of the speed of light, Eq. (34.1) must hold in any inertial reference frame. Consequently, if an interval equals zero in the frame K , it will be zero in any other frame K' .

Hence, an interval must become equal to zero in all reference frames simultaneously. It thus follows that the interval Δs between events expressed in the frame K must be related to the interval $\Delta s'$ between the same events expressed in the frame K' by the equation

$$\Delta s = a \Delta s' \quad (34.6)$$

But owing to the complete equivalence of the frames K and K' , we can write on the same grounds that

$$\Delta s' = a \Delta s \quad (34.7)$$

where a has the same value as in formula (34.6).

Multiplying relations (34.6) and (34.7), we find that

$$a^2 = 1$$

whence $a = \pm 1$. It is natural to assume that the sign of the interval in all the reference frames must be the same. Therefore the value of a equal to -1 must be discarded. We thus arrive at the conclusion that the interval between two events is an invariant:

$$\Delta s = \Delta s' \quad (34.8)$$

The result we have obtained indicates the expediency of the definition of the interval between two points of four-dimensional space that we have adopted. The interval determined by formula (34.3) is invariant with respect to transformations of the coordinates and time from one reference frame to another, i.e. behaves like the distance (34.2) between two points in conventional space.

We must underline the fact that the conclusion on the invariance of quantity (34.3) is a logical corollary of Einstein's postulates.

Proceeding from the invariance of the interval, we can write that

$$\Delta s^2 = c^2 \Delta t^2 - \Delta l^2 = c^2 \Delta t'^2 - \Delta l'^2 \quad (34.9)$$

Assume that $\Delta s^2 > 0$, i.e. that the interval is real. We can therefore find such a reference frame K' in which $\Delta l'$ will equal zero. In this frame, events separated by the interval Δs will occur at one point.

The time interval between events in the frame K' is

$$\Delta t' = \frac{\Delta s}{c} \quad (34.10)$$

Real intervals are called **time-like**.

Now assume that $\Delta s^2 < 0$, i.e. that the interval is imaginary. We can therefore find such a reference frame K' in which $\Delta t' = 0$, i.e. events occur simultaneously. The distance between the points at which the events occurred in the frame K' is

$$\Delta l' = i \Delta s \quad (34.11)$$

Imaginary intervals are called **space-like**.

Events occurring with the same particle can be separated only by a time-like interval. Indeed, since a particle cannot move at a velocity greater than c , the distance Δl which it travels during the time Δt cannot exceed $c \Delta t$, i.e. $\Delta l \leq c \Delta t$. Hence, $\Delta s^2 \geq 0$.

A space-like interval can separate only events having no causal relation. Indeed, if $\Delta s^2 < 0$, then $\Delta l > c \Delta t$. Consequently, no action emerging from one point of space can reach another point during the time Δt and affect an event occurring at this point.

Consider a particle moving uniformly at the velocity v relative to a frame K (a laboratory frame). Assume that two events occur with this particle and are separated by a time interval equal to dt in the frame K . Let us introduce a frame K' relative to which the particle is at rest. In this frame, the time interval between the events being considered is

$$dt' = \frac{ds}{c}$$

[see (34.10)].

It is a simple matter to see that the time interval dt' is measured by a clock moving together with the particle relative to K . The time measured by means of a clock moving together with a body is called the **proper time** of the body. Denoting the proper time by the symbol τ , we can write

$$d\tau = \frac{ds}{c} \quad (34.12)$$

Since ds is an invariant, and c is a constant, the proper time $d\tau$ is an invariant.

Let us find the relation between the proper time $d\tau$ and the time dt measured by means of a clock belonging to the frame K relative to which the particle and the (proper) clock associated with it move at the velocity v . For this purpose, we shall introduce into (34.12) the expression for ds in terms of the coordinates and the time in the

frame K :

$$d\tau = \frac{ds}{c} = \frac{\sqrt{c^2 dt^2 - dl^2}}{c}$$

[see formula (34.4)]. Let us transform the expression obtained as follows:

$$d\tau = dt \sqrt{1 - \frac{(dl/dt)^2}{c^2}}$$

But dl/dt is the velocity v of the particle. Hence,

$$d\tau = dt \sqrt{1 - \frac{v^2}{c^2}} \quad (34.13)$$

We conclude from (34.13) that the proper time of the particle is always less than the relevant time interval in the stationary (laboratory) frame.

We have obtained formula (34.13) for the uniform motion of a particle. It is also valid for non-uniform motion. We can therefore write the following equation for finite time intervals:

$$\Delta\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} dt \quad (34.14)$$

where $v = v(t)$ is the velocity of the body for which the proper time is being calculated.

35. Lorentz Transformations

We established in the preceding section that the interval Δs between two points in four-dimensional space is an invariant, i.e. behaves like the magnitude of a vector in Euclidean space. This gives us the grounds to consider Δs as the magnitude ("length") of a four-dimensional vector (a four-vector) conducted from one world point to another.

If we introduce the notation

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z \quad (35.1)$$

the square of the interval becomes

$$\Delta s^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$$

The following relation holds for the distance Δl between two points in Euclidean space:

$$\Delta l^2 = |\mathbf{r}_2 - \mathbf{r}_1|^2 = \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2$$

i.e. Δl equals the magnitude of the difference of the points' position vectors. Similarly, the interval Δs can be represented as the magnitude of the difference of four-position vectors of the relevant world

points. Consequently, the coordinates x^0, x^1, x^2, x^3 are the components of the four-position vector of a world point. The square of the magnitude of this position vector is

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

A comparison of the last expression with formula (XII.5) shows that the space in which an event is depicted by a world point with the coordinates (35.1) has a pseudo-Euclidean metric determined by the tensor (XII.4).

Consequently, the square of the four-position vector can be represented as

$$x^0x_0 + x^1x_1 + x^2x_2 + x^3x_3 = \sum_{\mu=0}^3 x^\mu x_\mu \quad (35.2)$$

[see formula (XII.31)].

The components of a four-position vector are transformed by the formula

$$x'^\mu = \sum_{\nu=0}^3 \alpha_\nu^\mu x^\nu \quad (35.3)$$

Let us take two inertial reference frames as the coordinate systems K and K' in a pseudo-Euclidean space. We shall direct the axes of these frames in accordance with Fig. 33.1. Hence, as is established in Appendix XII, the matrix of the transformation coefficients is as follows [see (XII.21)]:

$$[\alpha_\nu^\mu] = \begin{bmatrix} \alpha_0 & \alpha_1 & 0 & 0 \\ \alpha_1 & \alpha_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (35.4)$$

while

$$\alpha_0^2 - \alpha_1^2 = 1 \quad (35.5)$$

[see formula (XII.22)].

The coefficients α_0 and α_1 can depend only on the relative velocity v_0 of the frames. To find the form of this relation, let us write formula (35.3) for x'^1 . With a view to (35.4), we obtain

$$x'^1 = \alpha_1 x^0 + \alpha_0 x^1$$

We replace x'^1 with x' , x^0 with ct , and x^1 with x [see (35.1)]:

$$x' = \alpha_1 ct + \alpha_0 x \quad (35.6)$$

Let us write the expression obtained for point O' —the origin of coordinates of the frame K' (see Fig. 33.1). For this point, $x' = 0$,

and $x = v_0 t$. Substitution into (35.6) yields

$$0 = \alpha_1 c t + \alpha_0 v_0 t$$

whence

$$\alpha_1 = -\alpha_0 \frac{v_0}{c} \quad (37.5)$$

Introducing this value of α_1 into relation (35.5), we find that

$$\alpha_0^2 \left(1 - \frac{v_0^2}{c^2}\right) = 1$$

or

$$\alpha_0 = \frac{1}{\sqrt{1 - v_0^2/c^2}} \quad (35.8)$$

We obtain from (35.7) that

$$\alpha_1 = \frac{-v_0/c}{\sqrt{1 - v_0^2/c^2}} \quad (35.9)$$

Hence, the matrix (35.4) in the case we are interested in will be

$$[\alpha_\nu^\mu] = \begin{bmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{-\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ \frac{-\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (35.10)$$

We have introduced the symbol

$$\beta = \frac{v_0}{c} \quad (35.11)$$

The matrix $[\bar{\alpha}_\nu^\mu]$ of the reverse transformation differs from (35.10) only in that β in the numerator is preceded by a plus sign [see the matrix (XII.23)].

Substitution of the values of α_ν^μ we have found into (35.3) leads to formulas for the transformation of the components of a four-position vector:

$$x'^0 = \frac{x^0 - \beta x^1}{\sqrt{1-\beta^2}}, \quad x'^1 = \frac{-\beta x^0 + x^1}{\sqrt{1-\beta^2}}, \quad x'^2 = x^2, \quad x'^3 = x^3 \quad (35.12)$$

The formulas for the inverse transformation are

$$x^0 = \frac{x'^0 + \beta x'^1}{\sqrt{1-\beta^2}}, \quad x^1 = \frac{\beta x'^0 + x'^1}{\sqrt{1-\beta^2}}, \quad x^2 = x'^2, \quad x^3 = x'^3 \quad (35.13)$$

Going over in formulas (35.12) and (35.13) to the conventional symbols t, x, y, z , we obtain

$$t' = \frac{t - (v_0/c^2)x}{\sqrt{1-\beta^2}}, \quad x' = \frac{x - v_0 t}{\sqrt{1-\beta^2}}, \quad y' = y, \quad z' = z \quad (35.14)$$

$$t = \frac{t' + (v_0/c^2)x'}{\sqrt{1-\beta^2}}, \quad x = \frac{x' + v_0 t'}{\sqrt{1-\beta^2}}, \quad y = y', \quad z = z' \quad (35.15)$$

Formulas (35.14) and (35.15) are known as the Lorentz transformations. We invite our reader to convince himself that these transformations leave the interval s_{12} between two events invariant.

The formulas for the inverse transformation (35.15) differ from those for the direct transformation (35.14) only in the sign of v_0 . This should be expected with a view to the total equivalence of both reference frames, and also to the circumstance that for the given K and K' the projections of the velocity of relative motion onto the axes x and x' differ in their signs. Indeed, if the velocity of the frame K' relative to the frame K is directed to the right (and its projection onto the x -axis is positive), the velocity of the frame K relative to K' is directed to the left (and its projection onto the x' -axis is negative).

At velocities v_0 so small that the ratio v_0/c may be disregarded in comparison with unity, it is easy to see that the Lorentz transformations change into the Galilean ones.

The Lorentz transformations allow us to obtain formulas for the transformation of the lengths and time intervals in passing from one inertial reference frame to another (this is done in any general course of physics). We shall limit ourselves to recalling the formula for the Lorentz contraction of a body's length (in the direction of its motion):

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}} \quad (35.16)$$

Here l_0 is the proper length of the body, for instance a rod (i.e. the length of the body in the reference frame in which it is at rest), and l is the length of the body in the reference frame relative to which it is moving at the velocity v .

Since the lateral dimensions of a body do not change during its motion, the volume of the body contracts in accordance with the formula

$$V = V_0 \sqrt{1 - \frac{v^2}{c^2}} \quad (35.17)$$

where V_0 is the proper volume of the body, and V is its volume in the frame relative to which it is moving at the velocity v .

Let us find formulas for the transformation of the components of a particle's velocity. From formulas (35.14), we have

$$dt' = \frac{dt - (v_0/c^2) dx}{\sqrt{1 - \beta^2}}, \quad dx' = \frac{dx - v_0 dt}{\sqrt{1 - \beta^2}}, \quad dy' = dy, \quad dz' = dz$$

Consequently,

$$v'_x = \frac{dx'}{dt'} = \frac{dx - v_0 dt}{dt - (v_0/c^2) dx} = \frac{dx/dt - v_0}{1 - (v_0/c^2)(dx/dt)} = \frac{v_x - v_0}{1 - v_0 v_x / c^2}$$

Therefore,

$$v'_x = \frac{v_x - v_0}{1 - v_0 v_x / c^2} \quad (35.18)$$

Similar calculations lead to formulas for transforming the two other velocity components:

$$v'_y = \frac{v_y \sqrt{1 - \beta^2}}{1 - v_0 v_x / c^2}, \quad v'_z = \frac{v_z \sqrt{1 - \beta^2}}{1 - v_0 v_x / c^2} \quad (35.19)$$

The formulas for the inverse transformation are

$$\left. \begin{aligned} v_x &= \frac{v'_x + v_0}{1 + v_0 v'_x / c^2}, & v_y &= \frac{v'_y \sqrt{1 - \beta^2}}{1 + v_0 v'_x / c^2}, \\ v_z &= \frac{v'_z \sqrt{1 - \beta^2}}{1 + v_0 v'_x / c^2} \end{aligned} \right\} \quad (35.20)$$

Assume that the velocity \mathbf{v} of a particle makes the angle θ with the x -axis, and the angle θ' with the x' -axis (the axes x and x' , by parallel translation, can always be brought into a position in which they will be in the same plane as \mathbf{v}). Let us find the relation between the angles θ and θ' . We arrange the axes y and y' in the plane defined by the x -axis and the direction of the vector \mathbf{v} . The latter will therefore be in the plane xy , and we can write

$$v_x = v \cos \theta, \quad v_y = v \sin \theta$$

$$v'_x = v' \cos \theta', \quad v'_y = v' \sin \theta'$$

where v is the magnitude of the velocity in the frame K , and v' is its magnitude in the frame K' .

With the aid of formulas (35.18) and (35.19), we find that

$$\tan \theta' = \frac{v'_y}{v'_x} = \frac{v_y \sqrt{1 - \beta^2}}{v_x - v_0} = \frac{v \sin \theta \sqrt{1 - \beta^2}}{v \cos \theta - v_0} \quad (35.21)$$

This formula allows us to find the angle θ' made by the vector \mathbf{v}' with the x' -axis from the known values of v and θ . Similarly, we can find a formula allowing us to determine the angle θ between the vector \mathbf{v} and the x -axis when we know v' and θ' . Using formulas (35.20), we find that

$$\tan \theta = \frac{v_y}{v_x} = \frac{v'_y \sqrt{1 - \beta^2}}{v'_x + v_0} = \frac{v' \sin \theta' \sqrt{1 - \beta^2}}{v' \cos \theta' + v_0} \quad (35.22)$$

36. Four-Dimensional Velocity and Acceleration

In Appendix XII, we defined a four-vector as a set of the quantities a^0, a^1, a^2, a^3 which in passing over from one system of coordinates to another are transformed according to the same rules as the

components of a four-position vector. Consequently, the formulas for transforming the quantities a^μ are similar to formulas (35.12):

$$a'^0 = \frac{a^0 - \beta a^1}{\sqrt{1 - \beta^2}}, \quad a'^1 = \frac{-\beta a^0 + a^1}{\sqrt{1 - \beta^2}}, \quad a'^2 = a^2, \quad a'^3 = a^3 \quad (36.1)$$

The formulas for the inverse transformation differ from (36.1) in the sign of β :

$$a^0 = \frac{a'^0 + \beta a'^1}{\sqrt{1 - \beta^2}}, \quad a^1 = \frac{\beta a'^0 + a'^1}{\sqrt{1 - \beta^2}}, \quad a^2 = a'^2, \quad a^3 = a'^3 \quad (36.2)$$

Let us consider the four-vectors of velocity and acceleration. In non-relativistic mechanics, both the space intervals dl and the time intervals dt are assumed to be invariant. Therefore, the set of quantities obtained after dividing the components of the three-dimensional vector dr by the invariant dt forms the three-dimensional vector \mathbf{v} —the velocity vector of a particle. Similarly, the set of quantities obtained after dividing the components of the vector dv by the invariant dt is the acceleration vector \mathbf{w} .

We have seen that actually neither dl nor dt is invariant. What is invariant is the interval ds related to dl and dt by the expression $ds^2 = c^2 dt^2 - dl^2$. The invariance of the interval made it possible to introduce a four-position vector with the components x^0, x^1, x^2, x^3 that is an analogue of a three-dimensional position vector with the components x_1, x_2, x_3 . Let us attempt to find the four-dimensional analogues of the three-vectors \mathbf{v} and \mathbf{w} .

It is evident that the set of the four quantities dx^μ/dt does not have the properties of a four-vector because dt is not an invariant and $\sum (dx^\mu/dt) (dx_\mu/dt)$ does not retain its value in Lorentz transformations. But we know an invariant that is a "relative" of dt . It is the proper time $d\tau = ds/c$ [see (34.12)]. Since $d\tau$ is an invariant (i.e. a scalar), the quantities

$$u^\mu = \frac{dx^\mu}{d\tau} = c \frac{dx^\mu}{ds} \quad (36.3)$$

have the properties of components of a four-vector. It is called the four-dimensional velocity (four-velocity) of a particle¹.

Similarly, a four-vector with the components

$$w^\mu = \frac{d^2 x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} = c^2 \frac{du^\mu}{ds} \quad (36.4)$$

is called the four-dimensional acceleration of a particle.

¹ Einstein defined the four-velocity as a vector with the components $u^\mu = dx^\mu/ds$. It is obvious that the velocity defined in this way is a dimensionless quantity similar to v/c .

Taking into account the values of dx^μ , and also the circumstance that

$$d\tau = dt \sqrt{1 - \frac{v^2}{c^2}} \quad (36.5)$$

[see (34.13)], it is a simple matter to obtain the following values for the components of the four-velocity:

$$u^0 = \frac{c}{\sqrt{1 - v^2/c^2}}, \quad u^k = \frac{v_k}{\sqrt{1 - v^2/c^2}} \quad (k = 1, 2, 3) \quad (36.6)$$

which can be written as

$$u^\mu = \left(\frac{c}{\sqrt{1 - v^2/c^2}}, \frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} \right) \quad (36.7)$$

[see formulas (XII.34) and (XII.35)]. Here \mathbf{v} is the conventional three-dimensional velocity of a particle, and v_k are its projections onto the axes x, y, z . Of importance for our further discussion is the fact that when $v \ll c$, the spatial part of the four-velocity transforms into the conventional velocity \mathbf{v} .

We easily find from formulas (36.6) that

$$\sum_{\mu=0}^3 u^\mu u_\mu = c^2 \quad (36.8)$$

(if we determine u^μ in the same way as Einstein did, $\sum u^\mu u_\mu = 1$).

Differentiating formula (36.8) with respect to τ , we obtain

$$\sum_{\mu} \frac{du^\mu}{d\tau} u_\mu + \sum_{\mu} u^\mu \frac{du_\mu}{d\tau} = \sum_{\mu} w^\mu u_\mu + \sum_{\mu} u^\mu w_\mu = 0$$

According to (XII.33), both sums are equivalent, so that

$$\sum_{\mu} w^\mu u_\mu = 0 \quad (36.9)$$

It follows from Eq. (36.9) that the vectors of the four-dimensional velocity and acceleration are mutually perpendicular.

37. Relativistic Dynamics

Newton's equations are invariant with respect to the Galilean transformations, but are not invariant with respect to the Lorentz ones. Consequently, to satisfy Einstein's principle of relativity, Newton's second law must be replaced with a more general one. Having in view that when $\beta \rightarrow 0$ (i.e. when $v_0/c \rightarrow 0$), the Lorentz transformations convert to the Galilean transformations, the relativistic-invariant equations of motion at $v \ll c$ must convert to the

Newtonian equations:

$$\frac{d}{dt}(mv_i) = F_i \quad (i = 1, 2, 3) \quad (37.1)$$

The following relations are a natural four-dimensional generalization of these equations:

$$\frac{d}{d\tau}(mu^\mu) = K^\mu \quad (\mu = 0, 1, 2, 3) \quad (37.2)$$

where τ is the proper time, m is an invariant quantity characterizing the inert properties of a particle (the mass of the particle), u^μ is a component of the particle's four-velocity, and, finally, K^μ is a four-vector known as the Minkowski force. The values of K^μ must be determined so that when $v \ll c$, the spatial components of Eqs. (37.2) transform into Eqs. (37.1) like the spatial components of the four-velocity in this case transform into the conventional velocity v .

Taking into consideration expressions (36.5) and (36.6) for τ and u^μ , we shall write Eqs. (37.2) as

$$\begin{aligned} \frac{1}{\sqrt{1-v^2/c^2}} \frac{d}{dt} \left(\frac{mc}{\sqrt{1-v^2/c^2}} \right) &= K^0, \\ \frac{1}{\sqrt{1-v^2/c^2}} \frac{d}{dt} \left(\frac{mv_i}{\sqrt{1-v^2/c^2}} \right) &= K_i \quad (i = 1, 2, 3) \end{aligned}$$

Multiplying these equations by $\sqrt{1-v^2/c^2}$, we get

$$\frac{d}{dt} \left(\frac{mc}{\sqrt{1-v^2/c^2}} \right) = K^0 \sqrt{1-v^2/c^2} \quad (37.3)$$

$$\frac{d}{dt} \left(\frac{mv_i}{\sqrt{1-v^2/c^2}} \right) = K_i \sqrt{1-v^2/c^2} \quad (i = 1, 2, 3) \quad (37.4)$$

If we determine the spatial components K_i of the Minkowski force so that they are related to the components of the conventional three-dimensional force F_i by the expressions

$$F_i = K_i \sqrt{1-v^2/c^2} \quad (i = 1, 2, 3) \quad (37.5)$$

Eqs. (37.4) become

$$\frac{d}{dt} \left(\frac{mv_i}{\sqrt{1-v^2/c^2}} \right) = F_i \quad (i = 1, 2, 3) \quad (37.6)$$

It can be seen that when $v \ll c$, Eq. (37.6), as is required, transforms into Newton's equations (37.1).

To determine the time component K^0 of the Minkowski force, let us multiply Eq. (37.2) by the four-velocity u^μ . The result is

$$K^\mu u_\mu = m \frac{du^\mu}{d\tau} u_\mu = m w^\mu u_\mu$$

(we have taken into account that m is an invariant and, consequently, it can be put outside the derivative sign). Summation of the obtained equations over μ yields

$$\sum_{\mu=0}^3 K^{\mu} u_{\mu} = m \sum_{\mu=0}^3 w^{\mu} u_{\mu} = 0$$

[see (36.9)]. Substitution for u^{μ} of the values (36.6) and for K_i of the values obtained from (37.5) yields

$$K^0 \frac{c}{\sqrt{1-v^2/c^2}} - \sum_{i=1}^3 \frac{F_i}{\sqrt{1-v^2/c^2}} \frac{v_i}{\sqrt{1-v^2/c^2}} = 0$$

whence

$$K^0 = \frac{1}{c \sqrt{1-v^2/c^2}} \sum_{i=1}^3 F_i v_i = \frac{\mathbf{F} \mathbf{v}}{c \sqrt{1-v^2/c^2}} \quad (37.7)$$

Now we can write all the components of the Minkowski force. With a view to formulas (37.5) and (37.7), we have

$$K^0 = \frac{\mathbf{F} \mathbf{v}}{c \sqrt{1-v^2/c^2}}, \quad K^i = \frac{F_i}{\sqrt{1-v^2/c^2}} \quad (i=1, 2, 3) \quad (37.8)$$

Hence,

$$K^{\mu} = \left(\frac{\mathbf{F} \mathbf{v}}{c \sqrt{1-v^2/c^2}}, \quad \frac{\mathbf{F}}{\sqrt{1-v^2/c^2}} \right) \quad (37.9)$$

The scalar product of the three-vectors \mathbf{F} and \mathbf{v} gives the work done by the force \mathbf{F} on a particle in unit time. This work equals the rate of change in the particle's energy, i.e. dE/dt . Consequently, expression (37.7) for K^0 can be given the form

$$K^0 = \frac{1}{c \sqrt{1-v^2/c^2}} \frac{dE}{dt} \quad (37.10)$$

where E is the energy of a particle.

We have thus established that the relativistic-invariant¹ equation of the dynamics of a particle has the form of (37.2), where u^{μ} is the four-velocity with the components (36.6), and K^{μ} is the four-force (the Minkowski force) with the components (37.8). The spatial components of Eq. (37.2) can be represented in the form of (37.4) or (37.6). At the limit when $v \ll c$, these equations transform into Newton's equations [see (37.1)].

¹ Since u^{μ} and K^{μ} are four-vectors, the form of Eq. (37.2) in Lorentz transformations remains unchanged (m is an invariant by definition).

The time component of Eq. (37.2) [see (37.3)] after substituting for K^0 its value from (37.10) becomes

$$\frac{d}{dt} \left(\frac{mc}{\sqrt{1-v^2/c^2}} \right) = \frac{1}{c} \frac{dE}{dt} \quad (37.11)$$

We thus conclude that the relativistic expression for the energy of a particle is

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}} + \text{const} \quad (37.12)$$

38. Momentum and Energy of a Particle

In classical mechanics, the momentum of a particle is defined as a three-vector having the components

$$p_i^{(cl)} = mv_i \quad (i = 1, 2, 3) \quad (38.1)$$

The four-dimensional analogue of this momentum is the four-vector having the components

$$p^\mu = mu^\mu \quad (\mu = 0, 1, 2, 3) \quad (38.2)$$

where u^μ are the components of the four-velocity¹. Introducing the values (36.6) for u^μ , we obtain

$$p^0 = \frac{mc}{\sqrt{1-v^2/c^2}}, \quad p^i = \frac{mv_i}{\sqrt{1-v^2/c^2}} \quad (i = 1, 2, 3) \quad (38.3)$$

which can be written in the form

$$p^\mu = \left(\frac{mc}{\sqrt{1-v^2/c^2}}, \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} \right) \quad (38.4)$$

It is easy to see that when $v \ll c$, the formula for the spatial components of the relativistic momentum transforms into the Newtonian formula (38.1). This gives us the grounds to adopt the following formula as the relativistic expression for the conventional three-dimensional momentum:

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} \quad (38.5)$$

Let us now turn to the time component of the four-momentum. At the end of Sec. 37, we obtained formula (37.12) for the energy E of a particle without dealing with the value of the integration constant. Comparing expression (38.3) for p^0 with formula (37.12), it is a simple matter to see that by assuming the constant to be zero, we

¹ If the four-velocity was defined as $u^\mu = dx^\mu/ds$ (see the footnote on p. 135), the four-momentum is defined as $p^\mu = mc u^\mu$.

can obtain the relation

$$p^0 = \frac{E}{c} \quad (38.6)$$

In this case, the expression for the four-momentum becomes

$$p^\mu = \left(\frac{E}{c}, \mathbf{p} \right) \quad (38.7)$$

where \mathbf{p} is the quantity determined by formula (38.5).

The energy and (conventional) momentum are thus the components of a single four-vector—the four-momentum of a particle (this four-vector is sometimes called the momentum-energy four-vector). This circumstance allows us to use formulas (36.1) to find the rules for the transformation of E and \mathbf{p} in passing from one inertial reference frame to another. Introducing the relevant values of p^μ into (36.1), we easily obtain

$$E = \frac{E' + v_0 p'_x}{\sqrt{1 - \beta^2}}, \quad p_x = \frac{p'_x + v_0 (E'/c^2)}{\sqrt{1 - \beta^2}}, \quad p_y = p'_y, \quad p_z = p'_z \quad (38.8)$$

The inverse transformations differ in the sign of v_0 .

Let us find the square of the four-momentum. From Eq. (38.7), we get

$$\sum_{\mu=0}^3 p^\mu p_\mu = \frac{E^2}{c^2} - \mathbf{p}^2$$

[see formula (XII.38)]. At the same time

$$\sum_{\mu=0}^3 p^\mu p_\mu = \sum_{\mu=0}^3 (m u^\mu) (m u_\mu) = m^2 \sum_{\mu=0}^3 u^\mu u_\mu = m^2 c^2 \quad (38.9)$$

[see (36.8)]. We thus arrive at the relation

$$\frac{E^2}{c^2} - \mathbf{p}^2 = m^2 c^2 \quad (38.10)$$

We must note that the square of the four-momentum, like that of any four-vector, is an invariant.

Assuming the constant in formula (37.12) to be zero, we obtain the following expression for the energy of a particle:

$$E = \frac{m c^2}{\sqrt{1 - v^2/c^2}} \quad (38.11)$$

The quantity E determined by expression (38.11) is called the **total energy** of a particle. It must be borne in mind that E does not include the potential energy of the particle in an external force field¹.

¹ It must be remembered that by (37.7) and (37.10) $dE/dt = \mathbf{F} \cdot \mathbf{v}$. In Newtonian mechanics, on the other hand, the work of the resultant of all the forces acting on a particle equals the increment of its kinetic energy T , and not of its summary energy $T + U$.

For a particle at rest (i.e. when $v = 0$), expression (38.11) becomes

$$E_0 = mc^2 \quad (38.12)$$

where E_0 stands for the value of E at $v = 0$. This value is known as the **rest energy** of a particle. We remind our reader that for brevity's sake, we use the term particle to denote a point particle, i.e. a body whose dimensions we may disregard. The rest energy of such a body consists of the rest energies of the particles which the body consists of¹, of the kinetic energies of these particles, and of the energy of their interaction with one another. It thus follows that

$$mc^2 > \sum_a m_a c^2$$

where m is the mass of the body, and m_a are the rest masses of the particles forming the body. Hence, the mass of a body does not equal the sum of the masses of its parts.

Einstein at one time spent a lot of effort to substantiate the correctness of the assumption that the constant in (37.12) equals zero or, in other words, to substantiate the statement that the energy mc^2 is stored in the mass m [see (38.12)]. For this purpose, he considered several specific phenomena and showed that in each of them the change in a body's energy by ΔE leads to a change in its mass by $\Delta m = \Delta E/c^2$. Matters are much simpler at present. To substantiate the relation $\Delta E = c^2 \Delta m$, it is sufficient, for example, to consider the process of the transformation of an electron and a positron at rest into two gamma-quanta. The corresponding measurements show that the total energy of these gamma-quanta exactly equals the sum of the rest energies of the electron and the positron.

The difference between the total energy (38.11) and the rest energy (38.12) gives the kinetic energy of a particle

$$T = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 \quad (38.13)$$

At small v 's, this formula transforms into the Newtonian expression for the kinetic energy

$$T \approx mc^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2} \right) - mc^2 = \frac{1}{2} mv^2 \quad (38.14)$$

Let us now consider the momentum of a particle. Expression (38.5) which we have obtained can be interpreted to signify that the dependence of the momentum of a particle on the velocity is actually more complicated than is assumed in Newtonian mechanics. A different interpretation is also possible—we may consider that the relativistic momentum of a particle equals, as in Newtonian mechanics, the product of the mass and the velocity, but the mass is not constant

¹ We now give a different meaning to the term particle than we did up to now.

and depends on the velocity according to the law

$$m(v) = \frac{m}{\sqrt{1-v^2/c^2}} \quad (38.15)$$

Hence, for the relativistic momentum, we can write the expression

$$\mathbf{p} = m(v) \mathbf{v}$$

that is similar to the Newtonian expression $\mathbf{p} = m\mathbf{v}$.

Assuming in formula (38.15) that v equals zero, we find that $m(0) = m$. Consequently, the invariant quantity m can be considered as the value of the mass of a particle at rest. In general courses of physics, this quantity is called the rest mass and is usually designated by m_0 . The mass $m(v)$ determined by formula (38.15), on the other hand, is called the relativistic mass and is designated by m . It is customary practice in theoretical physics, however, to deal only with the invariant mass. There is therefore no need to use the subscript "0" on the symbol m and the word "rest" in the name of the mass of a particle.

The equation of motion (37.6) established in Sec. 37 can be written as follows:

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt} \left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} \right) = \mathbf{F} \quad (38.16)$$

It is not difficult to see that

$$\frac{d}{dt} \left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} \right) = \frac{m}{\sqrt{1-v^2/c^2}} \frac{d\mathbf{v}}{dt} + \frac{mv\mathbf{v}}{c^2 (1-v^2/c^2)^{3/2}} \frac{dv}{dt} \quad (38.17)$$

Assume that the velocity \mathbf{v} and the force \mathbf{F} acting on a particle are collinear. Now the velocity changes only in magnitude, and $\mathbf{v} (dv/dt) = v (dv/dt)$. After making this substitution in the second term of formula (38.17), putting dv/dt outside the parentheses, and performing simple transformations, we get

$$\frac{m}{(1-v^2/c^2)^{3/2}} \frac{d\mathbf{v}}{dt} = \mathbf{F} \quad (38.18)$$

Now assume that the force \mathbf{F} is perpendicular to the velocity \mathbf{v} . Hence, the velocity changes only in direction, and $dv/dt = 0$. Consequently, formula (38.16) becomes

$$\frac{m}{\sqrt{1-v^2/c^2}} \frac{d\mathbf{v}}{dt} = \mathbf{F} \quad (38.19)$$

A comparison of expressions (38.18) and (38.19) shows that the coefficient of proportionality between the force and the acceleration in these two cases is different (in Newtonian mechanics it has a single value equal to the mass of a particle).

In concluding, we shall note that the relation

$$E^2 = p^2 c^2 + m^2 c^4 \quad (38.20)$$

follows from formula (38.10). The energy expressed in terms of the momentum is called the Hamiltonian (see Sec. 30.). Consequently, the relativistic expression for the Hamiltonian of a particle is

$$H = c \sqrt{p^2 + m^2 c^2} \quad (38.21)$$

if the particle is free, and

$$H = c \sqrt{p^2 + m^2 c^2} + U \quad (38.22)$$

if the particle is in an external force field [U is the potential energy of the particle in this field; see the paragraph following formula (38.11)].

39. Action for a Relativistic Particle

Let us find the expression of the action for a free (i.e. not experiencing the action of any forces) particle. The integral expressing the action must be invariant relative to the Lorentz transformations. Consequently, it must be taken over a scalar, and the latter must have the form of a differential to the first power. The only scalar of this kind that can be associated with a free particle is a quantity proportional to the interval ds . Denoting the coefficient of proportionality by α , we get the following expression for the action:

$$S = \int_{t_1}^{t_2} \alpha ds = \int_{t_1}^{t_2} \alpha c \sqrt{1 - \frac{v^2}{c^2}} dt \quad (39.1)$$

[we have used formula (34.4) for ds^2 and taken into account that dl/dt equals the velocity v of the particle].

Comparing (39.1) with expression (7.1), we arrive at the conclusion that the Lagrangian for a free relativistic particle must be

$$L = \alpha c \sqrt{1 - \frac{v^2}{c^2}} \quad (39.2)$$

At the limit when $v \ll c$, this function must transform into the Newtonian expression

$$L = \frac{1}{2} m v^2 \quad (39.3)$$

Let us expand function (39.2) in powers of v/c . Ignoring the terms of the higher orders, we obtain

$$L = \alpha c \sqrt{1 - \frac{v^2}{c^2}} \approx \alpha c - \frac{\alpha v^2}{2c}$$

We may discard the constant term αc (see Sec. 7). Consequently, in the Newtonian approximation, $L = -\alpha v^2/2c$. A comparison with (39.3) shows that α must be assumed equal to $-mc$. We have thus established the form of the Lagrangian for a free particle¹:

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \quad (39.4)$$

Knowing the form of the Lagrangian, we can easily find the momentum and energy of a particle. Using formulas (9.5) and (5.1), we get

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \quad (39.5)$$

$$E = \mathbf{p}\mathbf{v} - L = \frac{mv^2}{\sqrt{1 - v^2/c^2}} + mc^2 \sqrt{1 - v^2/c^2} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \quad (39.6)$$

We have thus arrived at the same formulas for the momentum and energy of a particle that were obtained in Sec. 38. It must be borne in mind, however, that we obtained formulas (39.5) and (39.6) for a free particle, whereas in Sec. 38 similar formulas were obtained with the assumption that a particle experiences the action of forces.

Let us again revert to expression (39.1). With a view to the value of α we have found, we can write the action as

$$S = -mc \int_1^2 ds \quad (39.7)$$

The true trajectory of a particle is determined by the condition

$$\delta S = 0 \quad (39.8)$$

For the variation of the action, we have the expression

$$\delta S = -mc \delta \int_1^2 ds = -mc \int_1^2 \delta(ds)$$

The interval is

$$ds = \sqrt{\sum_{\mu} dx^{\mu} dx_{\mu}}$$

According to (XII.40), the variation of the radicand can be written as

$$\delta \sum_{\mu} dx^{\mu} dx_{\mu} = 2 \sum_{\mu} dx_{\mu} \delta dx^{\mu}$$

¹ For a particle in an external potential force field, we have

$$L = -mc^2 \sqrt{1 - v^2/c^2} - U$$

where U is the potential energy of the particle. We must note that in relativistic mechanics, L does not equal $T - U$.

Consequently,

$$\begin{aligned}\delta S &= -mc \int_1^2 \delta \sqrt{\sum_{\mu} dx^{\mu} dx_{\mu}} = -mc \int_1^2 \frac{\sum_{\mu} dx_{\mu} \delta dx^{\mu}}{\sqrt{\sum_{\mu} dx^{\mu} dx_{\mu}}} \\ &= -mc \int_1^2 \frac{\sum_{\mu} dx_{\mu} \delta dx^{\mu}}{ds} \quad (39.9)\end{aligned}$$

The derivative dx_{μ}/ds is u_{μ}/c [see (36.3)]. In addition, $\delta(dx^{\mu}) = d(\delta x^{\mu})$ [see (III.4)]. We therefore arrive at the expression

$$\delta S = -m \int_1^2 \sum_{\mu} u_{\mu} d(\delta x^{\mu})$$

Let us integrate this expression by parts:

$$\delta S = -m \sum_{\mu} u_{\mu} \delta x^{\mu} \Big|_1^2 + m \int_1^2 \sum_{\mu} \delta x^{\mu} \frac{du_{\mu}}{d\tau} d\tau \quad (39.10)$$

[we have represented du_{μ} in the form $(du_{\mu}/d\tau) d\tau$, where τ is the proper time of a particle]. At the ends of the trajectory, $\delta x^{\mu} = 0$. Therefore, condition (39.8) becomes

$$\delta S = m \int_1^2 \sum_{\mu} \delta x^{\mu} \frac{du_{\mu}}{d\tau} d\tau = 0 \quad (39.11)$$

For this condition to be observed with arbitrary values of δx^{μ} , it is essential that the quantities $du_{\mu}/d\tau$ vanish, i.e. the four-velocity of a particle be constant, which obviously holds for a free particle.

Let us find the action as a function of the coordinates of a particle (i.e. as a function of the upper limit of integration; see Sec. 32). For real motion, $du_{\mu}/d\tau = 0$, hence the second term in (39.10) vanishes. We consider that the lower limit of integration is fixed, therefore $(\delta x^{\mu})_1 = 0$. Consequently, the action as a function of the coordinates of a particle satisfies the relation

$$\delta S = -m \sum_{\mu=0}^3 u_{\mu} \delta x^{\mu}$$

(we have omitted the subscript "2" on δx^{μ}). The quantities mu_{μ} give covariant components of the four-momentum of a particle [see (38.2)] so that the increment of the action can be written as

$$\delta S = - \sum_{\mu=0}^3 p_{\mu} \delta x^{\mu} \quad (39.12)$$

In Sec. 32, we obtained the following expression for the increment of the action due to a change in the finite position of a particle in conventional (three-dimensional) space:

$$\delta S = \sum_{i=1}^3 p_i \delta x_i \quad (39.13)$$

[see formula (32.5)]. It is a simple matter to note that the terms of formula (39.12) corresponding to the spatial coordinates yield the sum (39.13) after the index on δx^μ has been lowered.

Examination of (39.12) shows that the covariant components of the four-momentum can be determined as follows:

$$p_\mu = -\frac{\partial S}{\partial x^\mu} \quad (39.14)$$

[compare with (32.6)].

It was established in Sec. 38 that $\sum p^\mu p_\mu = m^2 c^2$. Lowering the index of the first multiplier, i.e. replacing the contravariant multipliers with the corresponding covariant ones, we find that

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2 c^2$$

Substituting for p_μ in this equation their values from (39.14), we arrive at the Hamilton-Jacobi relativistic equation

$$\frac{1}{c^2} \left(\frac{\partial S}{\partial t} \right)^2 - \left(\frac{\partial S}{\partial x} \right)^2 - \left(\frac{\partial S}{\partial y} \right)^2 - \left(\frac{\partial S}{\partial z} \right)^2 = m^2 c^2 \quad (39.15)$$

(we have introduced ct instead of x^0 , x instead of x^1 , etc.).

The action S in (39.15) differs from the non-relativistic action S' . This can readily be understood if we have in mind that the action is related to the energy by the expression $E = -(\partial S / \partial t)$ [see formula (32.10)]. The non-relativistic energy E' , on the other hand, differs from the relativistic energy E in the term mc^2 (i.e. $E = E' + mc^2$). Hence

$$-\frac{\partial S}{\partial t} = -\frac{\partial S'}{\partial t} + mc^2$$

or

$$S = S' - mc^2 t$$

Using this relation in (39.15), we get an equation for S' :

$$\frac{1}{2mc^2} \left(\frac{\partial S'}{\partial t} \right)^2 - \frac{\partial S'}{\partial t} - \frac{1}{2m} \left[\left(\frac{\partial S'}{\partial x} \right)^2 + \left(\frac{\partial S'}{\partial y} \right)^2 + \left(\frac{\partial S'}{\partial z} \right)^2 \right] = 0$$

which at the limit when $c \rightarrow \infty$ transforms into the classical Hamilton-Jacobi equation for a free particle [see formula (32.15) in which we must assume that $U = 0$].

40. Energy-Momentum Tensor

In this section, we shall carry out a very important generalization of the expression for the action. In such a generalized form, the expression for the action can be applied not only to purely mechanical systems, but also to an electromagnetic field and other physical systems.

Up to now, we have written the expression for the action

$$S = \int_1^2 L(q_k, \dot{q}_k, t) dt \quad (40.1)$$

where L is the Lagrangian, $q_k = q_k(t)$ are the generalized coordinates determining the position of the particles of the system, and \dot{q}_k are the generalized velocities equal to dq_k/dt . The quantities q_k and \dot{q}_k are assumed to depend only on the time.

When we write equations in the four-dimensional form, we have to do with the four formally equivalent variables x^0, x^1, x^2, x^3 that must be present in equations in a similar way. To reflect this circumstance, we shall write the expression for the action as

$$S = \frac{1}{c} \int L^*(q_a, \dot{q}_{av}, x^0, x^1, x^2, x^3) dx^0 dx^1 dx^2 dx^3 \quad (40.2)$$

where by q_a we understand the set of the quantities q_1, q_2, \dots determining the state of a system (the parameters of the system). There may be any number of these parameters, particularly, an infinitely large one. By \dot{q}_{av} is understood the set of the partial derivatives of the parameters q_a with respect to the coordinates x^v :

$$\dot{q}_{av} = \frac{\partial q_a}{\partial x^v} \quad (a = 1, 2, \dots; v = 0, 1, 2, 3) \quad (40.3)$$

The factor $1/c$ has been introduced for convenience.

The quantities q_a and \dot{q}_{av} are considered as functions of the coordinates x^0, x^1, x^2, x^3 . Particularly, the parameters q_a may be found to depend only on x^0 . Here, we arrive at the case we already know when $q_a = q_a(t)$.

We must note that since $\partial^2 q_a / \partial x^v \partial x^\mu = \partial^2 q_a / \partial x^\mu \partial x^v$, the following relation holds:

$$\frac{\partial \dot{q}_{av}}{\partial x^\mu} = \frac{\partial \dot{q}_{a\mu}}{\partial x^v} \quad (40.4)$$

To establish the conformity between expressions (40.2) and (40.1), let us take into account that an element of the four-volume dV^* is related to an element of the volume dV in conventional space and

the time interval dt by the following expression:

$$dV^* = dx^0 dx^1 dx^2 dx^3 = c dt dV \quad (40.5)$$

Introducing this value of dV^* into (40.2), we get the expression

$$S = \int L^* dt dV = \int_1^2 dt \int L^* dV \quad (40.6)$$

Integration with respect to dt is performed within the given interval of time, with respect to dV —over the entire three-dimensional volume.

A comparison of expressions (40.1) and (40.6) shows that

$$L = \int L^* dV \quad (40.7)$$

Thus, the function L^* is the “density” of the Lagrangian of the system being considered.

For a closed mechanical system, the Lagrangian does not depend explicitly on t (see Sec. 8). Similarly, the absence of an explicit dependence of L^* on the coordinates x^0, x^1, x^2, x^3 should be a mathematical expression of the fact that the system is closed. Hence, for a closed system, the action has the form

$$S = \frac{1}{c} \int L^*(q_a, \dot{q}_{av}) dV^* \quad (40.8)$$

Let us find the equations of motion for a closed system. For this purpose, we shall calculate the variation of the action (40.8) and equate it to zero.

The variation of expression (40.8) is

$$\delta S = \frac{1}{c} \int \left(\sum_a \frac{\partial L^*}{\partial q_a} \delta q_a + \sum_{a,v} \frac{\partial L^*}{\partial \dot{q}_{av}} \delta \dot{q}_{av} \right) dV^*$$

By analogy with the relation $\delta y' = (\delta y)'$, we have

$$\delta \dot{q}_{av} = \delta \frac{\partial q_a}{\partial x^v} = \frac{\partial}{\partial x^v} \delta q_a$$

After performing this replacement, we obtain

$$\delta S = \frac{1}{c} \int \left(\sum_a \frac{\partial L^*}{\partial q_a} \delta q_a + \sum_{a,v} \frac{\partial L^*}{\partial \dot{q}_{av}} \frac{\partial}{\partial x^v} \delta q_a \right) dV^* \quad (40.9)$$

According to the rules for the differentiation of a product, we have

$$\sum_{a,v} \frac{\partial}{\partial x^v} \left(\frac{\partial L^*}{\partial \dot{q}_{av}} \delta q_a \right) = \sum_{a,v} \frac{\partial L^*}{\partial \dot{q}_{av}} \frac{\partial}{\partial x^v} \delta q_a + \sum_{a,v} \delta q_a \frac{\partial}{\partial x^v} \frac{\partial L^*}{\partial \dot{q}_{av}}$$

The first sum on the right-hand side is identical to the second term of the integrand in formula (40.9). We can therefore write

$$\delta S = \frac{1}{c} \int \left[\sum_a \frac{\partial L^*}{\partial q_a} \delta q_a + \sum_{a, v} \frac{\partial}{\partial x^v} \left(\frac{\partial L^*}{\partial \dot{q}_{av}} \delta q_a \right) - \sum_{a, v} \delta q_a \frac{\partial}{\partial x^v} \frac{\partial L^*}{\partial \dot{q}_{av}} \right] dV^* \quad (40.10)$$

The second of the sums

$$\sum_{a, v} \frac{\partial}{\partial x^v} \left(\frac{\partial L^*}{\partial \dot{q}_{av}} \delta q_a \right) = \sum_v \frac{\partial}{\partial x^v} \left(\sum_a \frac{\partial L^*}{\partial \dot{q}_{av}} \delta q_a \right)$$

is a four-divergence of a vector whose v -th component is

$$\sum_a \frac{\partial L^*}{\partial \dot{q}_{av}} \delta q_a$$

Therefore, using the four-dimensional analogue of the Ostrogradsky-Gauss theorem [see (XII.72)], the second of the three terms in formula (40.10) can be replaced with an integral over the closed hypersurface confining the four-volume over which integration is being performed in (40.10):

$$\int \sum_{a, v} \frac{\partial}{\partial x^v} \left(\frac{\partial L^*}{\partial \dot{q}_{av}} \delta q_a \right) dV^* = \oint \sum_v \left(\sum_a \frac{\partial L^*}{\partial \dot{q}_{av}} \delta q_a \right) df_v \quad (40.11)$$

On the boundary of the four-volume being considered, however, the variations $\delta q_a = 0$. (Similarly in mechanics, the variations δq_i at boundary points are zero.) Consequently, integral (40.11) vanishes, so that only the first and third sums remain in formula (40.10). Let us combine them, factoring out the common factor δq_a :

$$\delta S = \frac{1}{c} \int \sum_a \left[\frac{\partial L^*}{\partial q_a} - \sum_v \frac{\partial}{\partial x^v} \frac{\partial L^*}{\partial \dot{q}_{av}} \right] \delta q_a dV^*$$

Owing to the arbitrary nature of the variations δq_a , the expression we have obtained can equal zero only if all the expressions in brackets vanish. We thus arrive at the following equations of motion:

$$\frac{\partial L^*}{\partial q_a} = \sum_v \frac{\partial}{\partial x^v} \frac{\partial L^*}{\partial \dot{q}_{av}} \quad (a = 1, 2, \dots) \quad (40.12)$$

These equations are a generalization of Lagrange's equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \quad (40.13)$$

[see formula (7.3)]. The right-hand side of (40.12) is the sum of four derivatives because the role played by the single variable t is now played by four variables x^ν . It is not difficult to see that if q_a 's depend only on x^0 , i.e. on t , Eq. (40.12) transforms into Eq. (40.13).

Let us multiply Eq. (40.12) by $\dot{q}_{a\mu}$ and summate over a :

$$\sum_a \frac{\partial L^*}{\partial q_a} \dot{q}_{a\mu} = \sum_{a, \nu} \dot{q}_{a\mu} \frac{\partial}{\partial x^\nu} \frac{\partial L^*}{\partial \dot{q}_{a\nu}}$$

The right-hand side can be transformed by the formula

$$u \frac{\partial v}{\partial x^\nu} = \frac{\partial (uv)}{\partial x^\nu} - v \frac{\partial u}{\partial x^\nu}$$

The result is

$$\sum_a \frac{\partial L^*}{\partial q_a} \dot{q}_{a\mu} = \sum_{a, \nu} \frac{\partial}{\partial x^\nu} \left(\dot{q}_{a\mu} \frac{\partial L^*}{\partial \dot{q}_{a\nu}} \right) - \sum_{a, \nu} \frac{\partial L^*}{\partial \dot{q}_{a\nu}} \frac{\partial \dot{q}_{a\mu}}{\partial x^\nu} \quad (40.14)$$

Let us perform the substitutions $\dot{q}_{a\mu} = \partial q_a / \partial x^\mu$ in the sum on the left-hand side and $\partial \dot{q}_{a\mu} / \partial x^\nu = \partial^2 q_a / \partial x^\nu \partial x^\mu$ [see (40.4)] in the second sum on the right-hand side, and also group the terms of expression (40.14) in a different way. The result is the relation

$$\sum_a \frac{\partial L^*}{\partial q_a} \frac{\partial q_a}{\partial x^\mu} + \sum_{a, \nu} \frac{\partial L^*}{\partial \dot{q}_{a\nu}} \frac{\partial \dot{q}_{a\nu}}{\partial x^\mu} = \sum_{a, \nu} \frac{\partial}{\partial x^\nu} \left(\dot{q}_{a\mu} \frac{\partial L^*}{\partial \dot{q}_{a\nu}} \right)$$

The left-hand side of the expression obtained is $\partial L^* / \partial x^\mu$. Consequently, we have arrived at the formula

$$\frac{\partial L^*}{\partial x^\mu} = \sum_{a, \nu} \frac{\partial}{\partial x^\nu} \left(\dot{q}_{a\mu} \frac{\partial L^*}{\partial \dot{q}_{a\nu}} \right)$$

The left-hand side of this formula can be written as¹

$$\frac{\partial L^*}{\partial x^\mu} = \sum_\nu \delta_\mu^\nu \frac{\partial L^*}{\partial x^\nu} = \sum_\nu \frac{\partial}{\partial x^\nu} (\delta_\mu^\nu L^*)$$

which yields the relation

$$\sum_\nu \frac{\partial}{\partial x^\nu} (\delta_\mu^\nu L^*) = \sum_\nu \frac{\partial}{\partial x^\nu} \left(\sum_a \dot{q}_{a\mu} \frac{\partial L^*}{\partial \dot{q}_{a\nu}} \right)$$

The latter can be transformed as follows:

$$\sum_\nu \frac{\partial}{\partial x^\nu} \left[\sum_a \dot{q}_{a\mu} \frac{\partial L^*}{\partial \dot{q}_{a\nu}} - \delta_\mu^\nu L^* \right] = 0 \quad (40.15)$$

Equation (40.15) is a collection of four equations corresponding to different values of the index μ ($\mu = 0, 1, 2, 3$). The expressions

¹ It must be remembered that $\partial \varphi / \partial x^\nu$ is a covariant component of a four-vector.

in brackets have the properties of mixed components of a four-tensor of rank two. Denoting this tensor by the symbol $\tilde{T}_\mu{}^\nu$, we obtain:

$$\tilde{T}_\mu{}^\nu = \sum_a \dot{q}_{a\mu} \frac{\partial L^*}{\partial \dot{q}_{a\nu}} - \delta_\mu^\nu L^* \quad (40.16)$$

Using this symbol, we can write Eq. (40.15) in the form

$$\sum_\nu \frac{\partial \tilde{T}_\mu{}^\nu}{\partial x^\nu} = 0 \quad (\mu = 0, 1, 2, 3) \quad (40.17)$$

We remind our reader that we have obtained Eq. (40.15) by equating the variation of the action (40.8) to zero.

The tensor $\tilde{T}_\mu{}^\nu$ satisfying Eq. (40.17) is determined non-uniquely. Any tensor of the form

$$T_\mu{}^\nu = \tilde{T}_\mu{}^\nu + \sum_\rho \frac{\partial Q_\mu^{\nu\rho}}{\partial x^\rho} \quad (40.18)$$

where $Q_\mu^{\nu\rho}$ is a tensor antisymmetrical with respect to the indices ν and ρ ($Q_\mu^{\nu\rho} = -Q_\mu^{\rho\nu}$) also satisfies Eq. (40.17). Indeed, owing to the antisymmetry of the tensor $Q_\mu^{\nu\rho}$, we have

$$\frac{\partial^2 Q_\mu^{\nu\rho}}{\partial x^\nu \partial x^\rho} = - \frac{\partial^2 Q_\mu^{\rho\nu}}{\partial x^\rho \partial x^\nu}$$

and, consequently,

$$\sum_\nu \frac{\partial}{\partial x^\nu} \sum_\rho \frac{\partial Q_\mu^{\nu\rho}}{\partial x^\rho} = \sum_{\nu, \rho} \frac{\partial^2 Q_\mu^{\nu\rho}}{\partial x^\nu \partial x^\rho} = 0$$

The following condition will therefore follow from (40.17):

$$\sum_\nu \frac{\partial T_\mu{}^\nu}{\partial x^\nu} = 0 \quad (40.19)$$

By properly choosing the tensor $Q_\mu^{\nu\rho}$, we can always make the tensor (40.18) be symmetric. We shall assume in the following that this condition is observed and

$$T^{\mu\nu} = T^{\nu\mu} \quad (40.20)$$

We shall note that since $Q_\mu^{\nu\nu} = 0$,

$$T_\mu{}^\mu = \tilde{T}_\mu{}^\mu \quad (40.21)$$

i.e., $T^\mu{}_\mu$ is determined by formula (40.16).

The transition of the index μ upward in (40.19) will either leave all the T_{μ}^{ν} 's unchanged (if $\mu = 0$), or will reverse the sign of all the T_{μ}^{ν} 's (if $\mu = 1, 2, 3$). Consequently, it follows from (40.19) that

$$\sum_{\nu} \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0 \quad (\mu = 0, 1, 2, 3) \quad (40.22)$$

The tensor $T^{\mu\nu}$ is known as the **energy-momentum tensor** of a system. The grounds for this name will come to light below.

It is shown in Appendix XII that when the tensor $T^{\mu\nu}$ satisfies the condition (40.22) and all the $T^{\mu\nu}$'s vanish at infinity, the vector having the components

$$p^{\mu} = \alpha \int \sum_{\nu} T^{\mu\nu} df_{\nu} \quad (40.23)$$

[see formula (XII.86)], remains constant in time (is conserved). In Eq. (40.23), α is an arbitrary constant, and df_{ν} is a component of the four-vector of an element of the hypersurface. The integral in Eq. (40.23) is taken over an arbitrary hypersurface including the entire three-dimensional space. If we take the hyperplane $x^0 = \text{const}$ as the hypersurface over which integration is being performed, all the df_{ν} 's except $df_0 = dV$ vanish, and expression (40.23) will be simplified as follows:

$$p^{\mu} = \alpha \int T^{\mu 0} df_0 = \alpha \int T^{\mu 0} dV \quad (40.24)$$

We know that for a closed system the total (i.e. taken over the entire volume) energy and the total (i.e. taken over the entire volume) momentum are conserved. The energy and the momentum are the components of the four-momentum. Consequently, the four-momentum of a closed system must also be conserved. This gives us the grounds to identify the vector determined by formulas (40.23) and (40.24) with the four-momentum of a system. The constant α must be chosen so that formulas (40.23) and (40.24) agree with the previous definition of the four-momentum [see (38.7)], according to which, for example, $p^0 = E/c$. Assuming in formula (40.24) that $\mu = 0$, we get

$$p^0 = \alpha \int T^{00} dV = \frac{E}{c} \quad (40.25)$$

Using formulas (40.21) and (40.16), we can write

$$T^{00} = T_0^0 = \tilde{T}_0^0 = \sum_a \dot{q}_{a0} \frac{\partial L^*}{\partial \dot{q}_{a0}} - \delta_0^0 L^* = \sum_a \dot{q}_a \frac{\partial L^*}{\partial \dot{q}_a} - L^*$$

where $\dot{q}_a = dq_a/dt$ [we have taken into consideration that $\dot{q}_{a0} = \partial q_a / \partial x^0 = \partial q_a / \partial (ct) = (1/c) (\partial q_a / \partial t) = (1/c) \dot{q}_a$].

The integral $\int L^* dV$ gives the Lagrangian L . We therefore arrive at the relation

$$\int T^{00} dV = \sum_a \dot{q}_a \frac{\partial L}{\partial \dot{q}_a} - L$$

According to formula (5.1), the expression on the right-hand side determines the energy E of a system. Hence

$$\int T^{00} dV = E \quad (40.26)$$

and T^{00} is the energy density w :

$$T^{00} = w \quad (40.27)$$

Substitution of the value (40.26) into formula (40.25) leads us to the conclusion that $\alpha = 1/c$. Consequently,

$$p^\mu = \frac{1}{c} \int \sum_v T^{\mu v} df_v \quad (40.28)$$

if integration is performed over an arbitrary hypersurface including all the three-dimensional space, and

$$p^\mu = \frac{1}{c} \int T^{\mu 0} df_0 = \frac{1}{c} \int T^{\mu 0} dV \quad (40.29)$$

if integration is performed over the hyperplane $x^0 = \text{const.}$

To reveal the meaning of the components T^{0v} , let us write expression (40.22) for $\mu = 0$:

$$\sum_{v=0}^3 \frac{\partial T^{0v}}{\partial x^v} = \frac{\partial T^{00}}{\partial x^0} + \sum_{k=1}^3 \frac{\partial T^{0k}}{\partial x^k} = 0$$

or

$$-\frac{1}{c} \frac{\partial T^{00}}{\partial t} = \sum_{k=1}^3 \frac{\partial T^{0k}}{\partial x^k}$$

Multiplication of the relation obtained by c and integration over a certain volume V yield

$$-\frac{\partial}{\partial t} \int_V T^{00} dV = \int_V \sum_{k=1}^3 c \frac{\partial T^{0k}}{\partial x^k} dV \quad (40.30)$$

The integral on the left-hand side equals the energy confined in the volume V . The integrand on the right-hand side is the conventional (three-dimensional) divergence of a certain vector S having the components

$$S_x = cT^{01}, \quad S_y = cT^{02}, \quad S_z = cT^{03} \quad (40.31)$$

Formula (40.30) can thus be written as

$$-\frac{\partial E}{\partial t} = \int_V \nabla S \, dV$$

Applying the Ostrogradsky-Gauss theorem to the right-hand side, we obtain

$$-\frac{\partial E}{\partial t} = \oint_S S \, df$$

where the integral is taken over the closed surface confining the volume V .

The decrement of the energy in the volume V per unit time must equal the energy flux through the surface. Consequently, S is the vector of the energy flux density. By (40.31),

$$T^{01} = \frac{1}{c} S_x, \quad T^{02} = \frac{1}{c} S_y, \quad T^{03} = \frac{1}{c} S_z \quad (40.32)$$

Owing to symmetry of the tensor $T^{\mu\nu}$, the relation $T^{h0} = T^{0h}$ is observed, so that

$$T^{10} = \frac{1}{c} S_x, \quad T^{20} = \frac{1}{c} S_y, \quad T^{30} = \frac{1}{c} S_z \quad (40.33)$$

Hence, the components T^{0h} and T^{h0} to within the factor $1/c$ equal the relevant components of the vector of the energy flux density.

The spatial components of the vector (40.28) are

$$p^k = \frac{1}{c} \int T^{k0} \, dV$$

We thus conclude that the vector g with the components T^{h0}/c determines the density of the momentum of a system:

$$g_x = \frac{1}{c} T^{10}, \quad g_y = \frac{1}{c} T^{20}, \quad g_z = \frac{1}{c} T^{30} \quad (40.34)$$

A comparison of (40.34) and (40.33) shows that

$$g_k = \frac{1}{c^2} S_k \quad (40.35)$$

or in the vector form

$$g = \frac{d\mathbf{p}}{dV} = \frac{1}{c^2} \mathbf{S} \quad (40.36)$$

We have thus arrived at the conclusion that there is a relation between the energy flux and the momentum—the density of the momentum equals the density of the energy flux divided by c^2 .

To establish the meaning of the components T^{ik} , let us write relation (40.22) for $\mu = i$:

$$\sum_{v=0}^3 \frac{\partial T^{iv}}{\partial x^v} = \frac{\partial T^{i0}}{\partial x^0} + \sum_{k=1}^3 \frac{\partial T^{ik}}{\partial x^k} = 0$$

Hence, having in view that $x^0 = ct$ and $T^{i0} = cg_i$, we find that

$$-\frac{\partial g_i}{\partial t} = \sum_{k=1}^3 \frac{\partial T^{ik}}{\partial x^k}$$

Let us integrate the expression obtained over a certain volume V . Taking into account that $\int g dV = \mathbf{p}$ (here \mathbf{p} is the momentum of the part of the system confined in the volume V), we obtain

$$-\frac{\partial}{\partial t} p_i = \int_V \sum_{k=1}^3 \frac{\partial T^{ik}}{\partial x^k} dV$$

We transform the right-hand side according to the Ostrogradsky-Gauss theorem:

$$-\frac{\partial}{\partial t} p_i = \oint_f \sum_{k=1}^3 T^{ik} df_k = -\oint_f \sigma_i df \quad (40.37)$$

where σ_i is a vector with the components

$$\sigma_{ik} = T^{ik} \quad (40.38)$$

The left-hand side of formula (40.37) contains the rate of diminishing of the i -th component of the momentum confined in the volume V . Consequently, the right-hand side has the meaning of the flux of the component p_i through the surface f confining V , and σ_i is the density of the flux of p_i . The quantity (40.38) is the k -th component of the flux density of the component p_i . Hence, the three-dimensional tensor (40.38) determines the density of the momentum flux. We must note that the flux density of a scalar quantity (for instance, the energy) is a vector; the flux density of a vector quantity (for instance, the momentum) is a tensor.

The momentum carried through unit area in unit time equals the force acting on this area, i.e. the stress at the location of the area. This is why the tensor σ_{ik} is called the **stress tensor**.

We have thus established the meaning of all the components of the tensor $T^{\mu\nu}$. Combining (40.27), (40.32), (40.33), and (40.38),

we obtain

$$(T^{\mu\nu}) = \begin{pmatrix} w & S_x/c & S_y/c & S_z/c \\ S_x/c & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ S_y/c & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ S_z/c & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \quad (40.39)$$

The components of the tensor $T^{\mu\nu}$ characterize the density of the energy and the momentum, and also the densities of the fluxes of these quantities, which explains the name of $T^{\mu\nu}$ —the **energy-momentum tensor**.

Part Two

ELECTRODYNAMICS

Chapter VIII

ELECTROSTATICS

41. Electrostatic Field in a Vacuum

In general, an electric and a magnetic field are closely associated with each other, forming a single electromagnetic field. Stationary (i.e. not varying with time) electric and magnetic fields, however, may be treated separately. We shall begin with considering an electrostatic field in a vacuum.

The basic (force) characteristic of an electric field is the field strength \mathbf{E} related to the force acting on a point charge e at a given point in the field by the expression

$$\mathbf{F} = e\mathbf{E} \quad (41.1)$$

This relation may be considered as the definition of the quantity \mathbf{E} .

An electrostatic field is a potential one. This signifies that the work done by the forces of this field on a charge over any closed path is zero:

$$\oint \mathbf{F} d\mathbf{l} = e \oint \mathbf{E} d\mathbf{l} = 0 \quad (41.2)$$

A glance at formula (41.2) shows that the circulation of the electrostatic field strength vector over any contour (closed path) Γ is zero. Using Stokes's theorem [see (XI.23)], we can write

$$\oint_{\Gamma} \mathbf{E} d\mathbf{l} = \int_f [\nabla \mathbf{E}] d\mathbf{f} = 0 \quad (41.3)$$

where f is an arbitrary surface confined by the contour Γ , and $d\mathbf{f}$ is the vector of an elementary area taken on this surface.

Condition (41.3) must be satisfied for any arbitrarily chosen surface f . This is possible only if the integrand function at each point is zero. We thus arrive at the conclusion that the curl of an electrostatic field strength vector equals zero at every point of the field:

$$[\nabla \mathbf{E}] = 0 \quad (41.4)$$

The equality to zero of the strength curl is a feature of an electrostatic field expressing its potential nature.

It is known from vector analysis that the curl of the gradient of a scalar function always equals zero [see (XI.43)]. Therefore, the strength of an electrostatic field can be represented as the gradient of a scalar function φ :

$$\mathbf{E} = -\nabla\varphi \quad (41.5)$$

(the minus sign does not change matters, it has been taken from physical considerations). The function φ is known as the **potential of an electrostatic field**. It is obvious that φ has been determined to within an arbitrary constant addend. The potential can therefore be measured from any point of a field for which its value is taken equal to zero. In electrodynamics, the potential is usually assumed to be zero at infinity.

The potential of the field of a point charge e equals, as is known,

$$\varphi(\mathbf{r}) = \frac{e}{r} \quad (41.6)$$

where \mathbf{r} is a position vector drawn from the point where e is to the point for which φ is being determined.

Taking the gradient of expression (41.6) and reversing its sign, we find the field strength of a point charge:

$$\mathbf{E} = -\nabla\left(\frac{e}{r}\right) = \frac{e}{r^2} \frac{\mathbf{r}}{r} = \frac{e}{r^2} \mathbf{e}_r \quad (41.7)$$

where \mathbf{e}_r is the unit vector of the position vector \mathbf{r} [see (XI.51)].

Assume that a field is set up by a system of point charges e_a placed at points determined by the position vectors \mathbf{r}'_a (Fig. 41.1). Hence, according to the superposition principle, the potential of the field set up by this system at the point determined by the position vector \mathbf{r} is

$$\varphi(\mathbf{r}) = \sum_a \frac{e_a}{|\mathbf{r} - \mathbf{r}'_a|} \quad (41.8)$$

and the field strength is

$$\mathbf{E}(\mathbf{r}) = \sum_a \frac{e_a (\mathbf{r} - \mathbf{r}'_a)}{|\mathbf{r} - \mathbf{r}'_a|^3} \quad (41.9)$$

If the charge setting up a field is distributed in space with the density $\rho = \rho(\mathbf{r}')$, the potential and the strength of the field can be

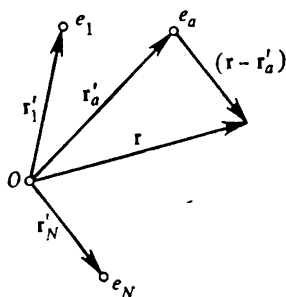


Fig. 41.1.

calculated by formulas similar to (41.8) and (41.9):

$$\varphi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|} \quad (41.10)$$

$$\mathbf{E}(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|^3} \quad (41.11)$$

where $dV' = dx' dy' dz'$ is a volume element at the point \mathbf{r}' (here x', y', z' are components of the varying vector \mathbf{r}').

The transition from the system of point charges e_a to a charge distributed in space with the density $\rho(\mathbf{r})$ is accomplished with the aid of the Dirac delta function (see Appendix XIII). This function allows us to represent the system of point charges e_a at points with the position vectors \mathbf{r}'_a by means of the charge density:

$$\rho(\mathbf{r}) = \sum_a e_a \delta(\mathbf{r} - \mathbf{r}'_a) \quad (41.12)$$

It is obvious that by introducing the function (41.12) into expressions (41.10) and (41.11) and performing integration, we shall arrive at formulas (41.8) and (41.9).

42. Poisson's Equation

The general course of physics acquaints us with Gauss's theorem, which for a field in a vacuum can be worded as follows: *the flux of the vector \mathbf{E} through a closed surface is proportional to the algebraic sum of the charges confined within the surface*. The proportionality constant depends on the choice of the system of units. In the Gaussian system usually employed in theoretical physics, it is 4π . Hence,

$$\oint_f \mathbf{E}_n df = 4\pi \sum e \quad (42.1)$$

If the distribution of charges inside the surface f is characterized with the aid of the charge density $\rho = \rho(\mathbf{r})$, Gauss's theorem can be written in the form

$$\oint_f \mathbf{E}_n df = 4\pi \int_V \rho dV \quad (42.2)$$

where V is the volume confined by the surface f .

Applying the Ostrogradsky-Gauss theorem [see (XI.13)] to the left-hand side of formula (42.2), we arrive at the relation

$$\int_V \nabla \mathbf{E} dV = 4\pi \int_V \rho dV$$

The latter expression must be observed for any arbitrarily selected volume V . This is possible only if at every point of the field

$$\nabla \mathbf{E} = 4\pi\rho \quad (42.3)$$

Expressing the field strength in formula (42.3) through the potential according to (41.5), we obtain

$$\nabla (\nabla\varphi) = -4\pi\rho$$

It is shown in Appendix XI [see (XI.38)] that $\nabla (\nabla\varphi) = \Delta\varphi$, where Δ is the Laplacian (operator). We thus arrive at the relation

$$\Delta\varphi = -4\pi\rho \quad (42.4)$$

known as **Poisson's equation**. In the expanded form, this equation is as follows:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = -4\pi\rho \quad (42.5)$$

Poisson's equation allows us to find the potential at every point of a field according to a given distribution of the charges in space (according to the given function ρ). Knowing φ , we can determine \mathbf{E} by formula (41.5).

For points of a field at which $\rho = 0$, Eq. (42.4) is

$$\Delta\varphi = 0 \quad (42.6)$$

Relation (42.6) is known as **Laplace's equation**.

In accordance with formula (41.10), the solution of Eq. (42.4) is

$$\varphi = \int \frac{\rho(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|} \quad (42.7)$$

where integration covers the entire region in which the charges setting up the field are distributed. This statement can be proved strictly mathematically by applying the Laplacian to the integral (42.7).

We must note that expression (42.7) also satisfies Poisson's equation if an arbitrary constant ($\Delta \text{ const} = 0$) is added to it. Consequently, generally speaking, for the solution of Poisson's equation to be fully unique, the boundary conditions must be set. Solution (42.7) (with $\text{const} = 0$) is obtained when the potential at infinity is assumed to be zero.

The solution of Poisson's equation can be shown to be the only one with given boundary conditions. This proof, however, is beyond the scope of the present text.

We shall show that the function $\varphi(\mathbf{r}) = 1/r$ satisfies Laplace's equation [Eq. (42.6)] at all points except $r = 0$. By (XI.51), we have $\nabla \frac{1}{r} = -\frac{1}{r^2} \mathbf{r} = -\frac{\mathbf{r}}{r^3}$. Applying the operator ∇ once more to this expression, we obtain

$$\Delta \frac{1}{r} = \nabla \left(\nabla \frac{1}{r} \right) = -\nabla \left(\frac{\mathbf{r}}{r^3} \right) = -\frac{1}{r^3} (\nabla \mathbf{r}) - \mathbf{r} \nabla \left(\frac{1}{r^3} \right)$$

The divergence of \mathbf{r} is 3 [see (XI.49)]. Consequently

$$\Delta \frac{1}{r} = -\frac{3}{r^3} - \mathbf{r} \left(-\frac{3}{r^4} \frac{\mathbf{r}}{r} \right) = -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

Q.E.D.

We can find the values of $\Delta (1/r)$ for all points including $r = 0$. To do this, we shall take advantage of the fact that for a point charge e at the origin of coordinates $\rho = e\delta(\mathbf{r})$, and the potential $\varphi = e/r$. Introducing these values into Eq. (42.4) and cancelling e , we arrive at the following mathematical relation:

$$\Delta \frac{1}{r} = -4\pi\delta(\mathbf{r}) \quad (42.8)$$

43. Expansion of a Field in Multipoles

Let us consider a system of charges in a restricted volume and investigate the field set up by this system at distances that are great in comparison with the system's dimensions. We shall place the origin of coordinates inside the volume occupied by the system. This gives the following expression for the potential:

$$\varphi(\mathbf{r}) = \sum_a \frac{e_a}{|\mathbf{r} - \mathbf{r}_a|} \quad (43.1)$$

where according to our assumption $r \gg r_a$ [see (41.8); we have omitted the prime on \mathbf{r}_a].

We expand expression (43.1) into a power series in \mathbf{r}_a/r . For this purpose, we express $\varphi(\mathbf{r})$ in terms of the components of the vectors \mathbf{r} and \mathbf{r}_a :

$$\varphi(x_1, x_2, x_3) = \sum_a \frac{e_a}{\sqrt{\sum_i (x_i - x_{ai})^2}} \quad (43.2)$$

(the subscript i takes on the values 1, 2, 3, and the subscript a , the values 1, 2, ..., n , where n is the number of charges in the system). Considering the quantities $(-x_{ai})$ as small increments of the coordinates x_i , we can write the function (43.2) as

$$\begin{aligned} \varphi(x_1, x_2, x_3) &= \sum_a \frac{e_a}{\sqrt{\sum_i x_i^2}} + \sum_a \sum_k \frac{\partial}{\partial x_k} \left(\frac{e_a}{\sqrt{\sum_i x_i^2}} \right) (-x_{ak}) \\ &+ \frac{1}{2} \sum_a \sum_{k, m} \frac{\partial^2}{\partial x_k \partial x_m} \left(\frac{e_a}{\sqrt{\sum_i x_i^2}} \right) (-x_{ak}) (-x_{am}) + \dots \\ &= \varphi_0 + \varphi_1 + \varphi_2 + \dots \end{aligned} \quad (43.3)$$

[each addend of the sum (43.2) is expanded into a series].

Expression (43.3) is known as the expansion of the potential in multipoles. The first term of the expansion

$$\varphi_0 = \frac{\sum e_a}{r} \quad (43.4)$$

(we have substituted r for $\sqrt{\sum x_i^2}$) has the form of the potential of a point charge. The total charge $\sum e_a$ is a zero-order multipole (it is also called a monopole). When this multipole is non-zero, the term φ_0 makes the main contribution to potential (43.2).

To establish the form of a first-order multipole, let us transform the second term of the expansion (43.3) as follows:

$$\varphi_1 = \sum_a \sum_k \frac{\partial}{\partial x_k} \left(\frac{e_a}{r} \right) (-x_{ak}) = - \sum_k \frac{\partial}{\partial x_k} \left(\frac{1}{r} \right) \sum_a e_a x_{ak}$$

The sum $\sum_a e_a x_{ak}$ is the projection onto the k -th coordinate axis of the vector

$$\mathbf{p} = \sum_a e_a \mathbf{r}_a \quad (43.5)$$

This vector is the dipole moment of the system of charges¹. And it is a first-order multipole.

The expression

$$\frac{\partial}{\partial x_k} \left(\frac{1}{r} \right)$$

gives the k -th component of the gradient of $1/r$. Hence,

$$\varphi_1 = - \sum_k \left(\nabla \frac{1}{r} \right)_{\text{pr. } x_k} \cdot p_k = - \mathbf{p} \cdot \nabla \frac{1}{r} \quad (43.7)$$

The formula for φ_1 can be obtained directly in the vector form, taking advantage of the circumstance that by (XI.5) to within first-order terms, we have

$$f(\mathbf{r} + \delta \mathbf{r}) = f(\mathbf{r}) + \nabla f(\mathbf{r}) \cdot \delta \mathbf{r}$$

¹ If a charge is distributed over the volume of a system having the density ρ , the dipole moment is determined by the integral

$$\mathbf{p} = \int \rho(\mathbf{r}) \mathbf{r} dV \quad (43.6)$$

It must be remembered that for a system whose total charge $\sum e_a$ equals zero, the dipole moment does not depend on the choice of the origin of coordinates. Indeed, transferring the origin of coordinates to a point for which $\mathbf{r} = \mathbf{b}$, we get the values $\mathbf{r}'_a = \mathbf{r}_a - \mathbf{b}$ for the position vectors from the new origin of coordinates. The dipole moment in the new system is $\mathbf{p}' = \sum e_a \mathbf{r}'_a = \sum e_a \mathbf{r}_a - \sum e_a \mathbf{b} = \mathbf{p} - \mathbf{b} \sum e_a$. Since $\sum e_a = 0$, we obtain $\mathbf{p}' = \mathbf{p}$.

Let us apply this formula to each addend in (43.1), considering $-r_a$ as δr . The result is

$$\varphi(\mathbf{r}) = \sum_a \frac{e_a}{r} - \sum_a e_a r_a \nabla \frac{1}{r} \quad (43.8)$$

The first term coincides with (43.4), and the second with (43.7).

We must note that $\nabla(1/r)$ is proportional to $1/r^2$. Consequently, the addends of the second sum in (43.8) are quantities of the order of r_a/r relative to the addends of the first sum.

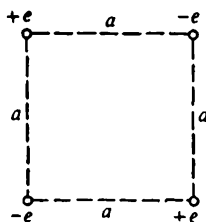


Fig. 43.1.

After calculating $\nabla(1/r)$ and introducing it into (43.7), we arrive at an expression for the potential of the field of a dipole:

$$\varphi_1 = -\mathbf{p} \nabla \frac{1}{r} = -\mathbf{p} \left(-\frac{1}{r^2} \frac{\mathbf{r}}{r} \right) = \frac{\mathbf{p} \mathbf{r}}{r^3} \quad (43.9)$$

Now let us find the field strength of a dipole:

$$\mathbf{E} = -\nabla \varphi_1 = -\nabla \left(\frac{\mathbf{p} \mathbf{r}}{r^3} \right) = -(\mathbf{p} \mathbf{r}) \nabla \frac{1}{r^3} - \frac{1}{r^3} \nabla (\mathbf{p} \mathbf{r})$$

[see (XI.25)]. Using formulas (XI.51) and (XI.37), we obtain

$$\mathbf{E} = \frac{3(\mathbf{p} \mathbf{r})}{r^4} \frac{\mathbf{r}}{r} - \frac{1}{r^3} \{ [\mathbf{p}, [\nabla \mathbf{r}]] + [\mathbf{r}, [\nabla \mathbf{p}]] + (\mathbf{r} \nabla) \mathbf{p} + (\mathbf{p} \nabla) \mathbf{r} \}$$

The curl of \mathbf{r} equals zero [see (XI.50)], $[\nabla \mathbf{p}] = 0$ because \mathbf{p} does not depend on \mathbf{r} , and $(\mathbf{r} \nabla) \mathbf{p} = 0$ for the same reason. Hence, of the four terms in the braces, only the last one is non-zero, and by (XI.34) it equals \mathbf{p} . Therefore,

$$\mathbf{E} = \frac{3(\mathbf{p} \mathbf{r})}{r^4} \frac{\mathbf{r}}{r} - \frac{\mathbf{p}}{r^3} = \frac{3\mathbf{e}_r (\mathbf{p} \mathbf{e}_r) - \mathbf{p}}{r^3} \quad (43.10)$$

where \mathbf{e}_r is the unit vector of the position vector \mathbf{r} .

We must note that the field of a dipole has axial symmetry relative to the direction of \mathbf{p} .

Not only the total charge $\sum e_a$, but also the dipole moment $\mathbf{p} = \sum e_a \mathbf{r}_a$ may equal zero. This occurs, for example, for the system of charges depicted in Fig. 43.1 and known as a **quadrupole**. Here,

the field is determined by the next term of the expansion of the function (43.1), quadratic relative to the quantities \mathbf{r}_a/r .

Let us write φ_2 as follows [see (43.3)]:

$$\begin{aligned}\varphi_2 &= \frac{1}{2} \sum_a \sum_{k, m} \frac{\partial^2}{\partial x_k \partial x_m} \left(\frac{e_a}{r} \right) x_{ak} x_{am} \\ &= \frac{1}{2} \sum_{k, m} \frac{\partial^2}{\partial x_k \partial x_m} \left(\frac{1}{r} \right) \left\{ \sum_a e_a x_{ak} x_{am} \right\} \quad (43.11)\end{aligned}$$

The quantity in braces is the (k, m) -th component of a symmetrical tensor of rank two (see Appendix X). This tensor could be adopted as the corresponding multipole. But as we shall show below, of the nine components of this tensor not six are independent (as in a symmetrical tensor), but only five. To underline this, the tensor characterizing the property of the system that determines φ_2 is written in a different form. We saw in Sec. 42 that the function $1/r$ satisfies Laplace's equation, i.e.

$$\Delta \frac{1}{r} = \sum_k \frac{\partial^2}{\partial x_k^2} \frac{1}{r} = 0$$

(according to our assumption $r \gg r_a$, so that $r = 0$ is not considered). It can easily be seen that this formula can be written as

$$\sum_{k, m} \delta_{km} \frac{\partial^2}{\partial x_k \partial x_m} \frac{1}{r} = 0$$

Multiplying the expression we have obtained by $e_a r_a^2/6$ and then summing over a , we obtain

$$\frac{1}{6} \sum_a e_a r_a^2 \sum_{k, m} \delta_{km} \frac{\partial^2}{\partial x_k \partial x_m} \frac{1}{r} = 0$$

which can be written as follows:

$$\frac{1}{2} \sum_{k, m} \frac{\partial^2}{\partial x_k \partial x_m} \frac{1}{r} \left\{ \sum_a \frac{1}{3} e_a r_a^2 \delta_{km} \right\} = 0$$

Subtracting this expression from (43.11), we give the formula for φ_2 the form

$$\varphi_2 = \frac{1}{2} \sum_{k, m} \frac{\partial^2}{\partial x_k \partial x_m} \frac{1}{r} \left\{ \sum_a e_a \left(x_{ak} x_{am} - \frac{1}{3} r_a^2 \delta_{km} \right) \right\} \quad (43.12)$$

The set of quantities

$$Q_{km} = \sum_a e_a (3x_{ak} x_{am} - r_a^2 \delta_{km}) \quad (43.13)$$

is called the tensor of the quadrupole moment of a system. Let us evaluate the trace of this tensor, i.e. the sum of its diagonal compo-

nents:

$$\begin{aligned}\text{Tr}(Q_{km}) &= \sum_k Q_{kk} = \sum_k \sum_a e_a (3x_{ak}^2 - r_a^2) = \sum_a e_a \sum_k (3x_{ak}^2 - r_a^2) \\ &= \sum_a e_a \left\{ 3 \sum_k x_{ak}^2 - 3r_a^2 \right\} = 0\end{aligned}\quad (43.14)$$

The equality to zero of $\text{Tr}(Q_{km})$ signifies that of the three diagonal components of the tensor Q_{km} only two are independent, and, consequently, there are altogether five independent components.

If we transform the tensor Q_{km} to the principal axes, owing to condition (43.14) only two of the three principal values are independent. If a system of charges has an axis of symmetry of an order higher than two, this axis (we shall designate it by the letter z) is one of the principal axes of the tensor Q_{km} ; the position of the other two principal axes is arbitrary. In this case, it is evident that $Q_{xx} = Q_{yy}$, and owing to (43.14), we have

$$Q_{xx} = Q_{yy} = -\frac{1}{2} Q_{zz} \quad (43.15)$$

The principal value Q_{zz} is called simply the quadrupole moment of the system in this case. It can be shown that when the total charge and the dipole moment of a system equal zero, the quadrupole moment does not depend on the choice of the origin of coordinates.

Using the symbol (43.13), we can write the potential of a field due to a quadrupole as follows:

$$\varphi_2 = \frac{1}{6} \sum_{k, m} Q_{km} \frac{\partial^2}{\partial x_k \partial x_m} \frac{1}{r} \quad (43.16)$$

Let us calculate the second derivatives in this expression:

$$\begin{aligned}\frac{\partial^2}{\partial x_k \partial x_m} \frac{1}{r} &= \frac{\partial}{\partial x_k} \left(\frac{\partial}{\partial x_m} \frac{1}{r} \right) = \frac{\partial}{\partial x_k} \left(-\frac{1}{r^2} \frac{\partial r}{\partial x_m} \right) \\ &= \frac{\partial}{\partial x_k} \left(-\frac{x_m}{r^3} \right) = -\frac{\delta_{km}}{r^3} + \frac{3x_m x_k}{r^5}\end{aligned}$$

($\partial x_m / \partial x_k = \delta_{km}$). Introducing this value of the derivatives into (43.16), we have

$$\varphi_2 = \frac{1}{6} \sum_{k, m} \frac{Q_{km}}{r^3} \left(\frac{3x_m x_k}{r^2} - \delta_{km} \right) \quad (43.17)$$

Examination of (43.17) shows that the potential of a quadrupole diminishes with the distance as $1/r^3$. We remind our reader that the potential of a monopole diminishes according to the law $1/r$ [see (43.4)], and that of a dipole according to the law $1/r^2$ [see (43.9)]. In general, the potential of an n -th order multipole diminishes with the distance according to the law $1/r^{n+1}$.

Hence, the field of a system of charges can be represented as the superposition of the fields set up by multipoles of different orders:

$$\varphi(\mathbf{r}) = \varphi_0 + \varphi_1 + \varphi_2 + \dots = \frac{\sum e_a}{r} - p \nabla \frac{1}{r} + \frac{1}{6} \sum Q_{km} \frac{\partial^2}{\partial x_k \partial x_m} \frac{1}{r} + \dots$$

We shall not deal with multipoles of higher orders. We shall only note that a multipole of the third order is called an **octupole**. We can indicate as an example of an octupole a system of eight unlike charges identical in magnitude and arranged at the corners of a cube so that the closest neighbours are charges of opposite signs.

44. Field in Dielectrics

Up to now, we have been dealing with an electrostatic field produced by a given system of charges in a vacuum. It was assumed that the charges setting up the field can move over macroscopic distances (for instance, within the confines of the entire conducting body). We shall call such charges **free**.

Matters become much more complicated in a field produced by free charges in dielectrics. Here the field set up by the charges in the atoms and molecules of the dielectric is superposed onto the field of the free charges. Since these charges cannot leave the confines of the atoms and molecules they belong to, they are called **bound**.

If \mathbf{E}_{free} is the field of the free charges, and $\mathbf{E}_{\text{bound}}$ is that of the bound ones, the strength of the resultant field \mathbf{E}_{res} can be written as

$$\mathbf{E}_{\text{res}} = \mathbf{E}_{\text{free}} + \mathbf{E}_{\text{bound}} \quad (44.1)$$

Even if the free charges are stationary, the field (44.1) is not stationary (i.e. time independent) because the bound charges are in motion inside the molecules and, in addition, participate together with the molecules in thermal motion. It is thus evident that the field $\mathbf{E}_{\text{bound}}$ is a fast-varying function of time. In addition, $\mathbf{E}_{\text{bound}}$ changes greatly in the space between two adjacent molecules. Both kinds of dependence (on the time and on the point in the space between molecules) vanish if we deal with a value of $\mathbf{E}_{\text{bound}}$ that has been averaged, first, over a time interval much longer than the period of intramolecular motion and thermal oscillations, and, second, over a volume considerably exceeding that of a molecule. Consequently, the field $\langle \mathbf{E}_{\text{bound}} \rangle$ is stationary. In addition, it changes smoothly within the limits of a volume including many molecules. The field $\langle \mathbf{E}_{\text{bound}} \rangle$ is called macroscopic, unlike the microscopic field $\mathbf{E}_{\text{bound}}$.

We shall call the macroscopic quantity

$$\mathbf{E} = \mathbf{E}_{\text{free}} + \langle \mathbf{E}_{\text{bound}} \rangle \quad (44.2)$$

the field strength in a dielectric.

In the absence of an external field (i.e. a field of free charges), the field $\langle \mathbf{E}_{\text{bound}} \rangle$ usually equals zero. Under the action of an external field, the mean positions of bound charges are displaced the more, the stronger is the field acting on them. As a result, the field $\langle \mathbf{E}_{\text{bound}} \rangle$ becomes other than zero. In calculating the field (44.2), matters become complicated by the fact that the average displacement of the bound charges is determined not by the field \mathbf{E}_{free} , but by the resultant field \mathbf{E} that includes $\langle \mathbf{E}_{\text{bound}} \rangle$ in it.

It is customary practice to characterize the state of a dielectric by the dipole moment of a unit of volume of the dielectric, which is called the **polarization** and is designated by the symbol \mathbf{P} . It is evident that \mathbf{P} can be determined as

$$\mathbf{P} = \frac{\sum \mathbf{p}_i}{\Delta V} \quad (44.3)$$

where ΔV is an infinitely small volume¹, and \mathbf{p}_i is the dipole moment of an individual molecule, summation is performed over all the molecules confined in the volume ΔV .

Having determined \mathbf{P} in this way, we have in essence performed the averaging mentioned above when discussing $\mathbf{E}_{\text{bound}}$ (\mathbf{P} is a macroscopic quantity, and \mathbf{p}_i is a microscopic one).

When an electric field acts on a dielectric, the bound charges become displaced (each remaining within the confines of its "own" molecule), the positive ones along the field, and the negative ones in the opposite direction. The result is the formation of bound charges on the surface of the dielectric. In addition, space bound charges may also appear.

Let us find an expression for ρ_{bound} — the volume density of the bound charges. We mentally separate in a dielectric an infinitely small volume ΔV . Assume that in the absence of a field the bound charges e_α (where $\alpha = 1, 2, \dots$) confined in this volume are at points determined by the position vectors $\mathbf{r}_{\alpha 0}$. Since the dielectric is not polarized in the absence of a field, the expression $\sum e_\alpha \mathbf{r}_{\alpha 0}$ is zero (the sum is taken over ΔV). Indeed, with an accuracy up to $1/\Delta V$, this sum coincides with the polarization \mathbf{P} , which vanishes in an unpolarized dielectric.

¹ By an infinitely small volume in physics is meant the volume ΔV containing a sufficiently large number of molecules to allow us to ignore the discreteness of the substance, and at the same time small enough for us to consider that macroscopic quantities such as \mathbf{E} or \mathbf{P} are constant within the confines of ΔV .

Assume that when a field is switched on, the bound charges become displaced by the segments $\Delta \mathbf{r}_\alpha$ (we must note that these segments are much smaller than the linear dimensions of the volume ΔV). The result is the appearance of a polarization characterized by the vector

$$\mathbf{P} = \frac{1}{\Delta V} \sum_{\alpha} e_{\alpha} (\mathbf{r}_{\alpha 0} + \Delta \mathbf{r}_{\alpha}) = \frac{1}{\Delta V} \sum_{\alpha} e_{\alpha} \Delta \mathbf{r}_{\alpha} \quad (44.4)$$

The bound charges in a dielectric can be divided into several groups, in each of which the magnitude of the charge e_{α} and the displacement $\Delta \mathbf{r}_{\alpha}$ are identical. Let us number such groups with the subscript β . Let n_{β} stand for the number of charges of group β per unit volume of the dielectric. Hence, the polarization can be written as follows:

$$\mathbf{P} = \frac{1}{\Delta V} \sum_{\beta} e_{\beta} \Delta \mathbf{r}_{\beta} n_{\beta} \Delta V = \sum_{\beta} n_{\beta} e_{\beta} \Delta \mathbf{r}_{\beta} \quad (44.5)$$

($n_{\beta} \Delta V$ is the number of charges of group β contained in the volume ΔV).

We calculate the algebraic sum of the bound charges intersecting the boundary of the volume ΔV when the field is switched on. Figure 44.1 shows an element df of the surface confining ΔV . The charges of the group bearing the number β contained in an elementary volume of the size $\Delta \mathbf{r}_{\beta} df$ cross df and emerge from or enter the elementary volume. They carry along with them the charge

$$n_{\beta} e_{\beta} \Delta \mathbf{r}_{\beta} df \quad (44.6)$$

This expression is algebraic. Its sign depends on that of e_{β} and on the sign of the scalar product $\Delta \mathbf{r}_{\beta} d\mathbf{f}$, i.e. on the direction of $\Delta \mathbf{r}_{\beta}$ relative to the outward normal to df (in Fig. 44.1 it is denoted by \mathbf{n}).

Summing expression (44.6) over β , we obtain the total charge crossing df :

$$\sum_{\beta} n_{\beta} e_{\beta} \Delta \mathbf{r}_{\beta} d\mathbf{f} = \mathbf{P} d\mathbf{f} \quad (44.7)$$

[see (44.5)].

Having integrated expression (44.7) over the surface f , let us find the total bound charge emerging from the volume ΔV when a field is switched on. When this happens, the volume ΔV , which was previously neutral, acquires a bound charge:

$$\Delta q_{\text{bound}} = - \int_f \mathbf{P} d\mathbf{f}$$

(the acquired charge equals the emerged one taken with the opposite sign). Using the Ostrogradsky-Gauss theorem, we obtain

$$\Delta q_{\text{bound}} = - \int_{\Delta V} \nabla P \, dV = - \nabla P \, \Delta V$$

Hence, the following expression is obtained for the density of bound charges:

$$\rho_{\text{bound}} = -\nabla P \quad (44.8)$$

Formula (44.8) will also help us to find the surface density of the bound charges σ_{bound} . For this purpose, let us consider a cylindrical volume confined between two infinitely close surfaces of the magnitude Δf at different sides of the dielectric's surface. (Fig. 44.2). The bound charge q_{bound} confined in this volume can be represented either in the form $\int \rho_{\text{bound}} \, dV$, or in the form $\sigma_{\text{bound}} \Delta f$. Using formula (44.8) and the Ostrogradsky-Gauss theorem, we can write

$$q_{\text{bound}} = \sigma_{\text{bound}} \Delta f = - \int \nabla P \, dV = - \int P_{n'} \, df$$

where n' is an outward normal to the surface of the cylinder. The integral over the surface can be divided into three parts: an integral over the outer base of the cylinder that equals zero because $P = 0$ outside the dielectric; an integral over the side surface of the cylinder that may be disregarded owing to this surface being infinitely small, and, finally, an integral over the inner base of the cylinder. The latter equals $-P_{n'} \Delta f$, where n' is an inward normal to the surface of the dielectric. If instead of n' we take the outward normal n , we must substitute $+P_n$ for $-P_{n'}$. We have thus arrived at the expression $\sigma_{\text{bound}} \Delta f = P_n \Delta f$, whence

$$\sigma_{\text{bound}} = P_n \quad (44.9)$$

Here P_n is the projection of the polarization P onto an outward normal to the surface of a dielectric.

We must note that charges of the density determined by formulas (44.8) and (44.9) are not imaginary, but quite real charges.

Hence, in the presence of dielectrics, the field of bound surface [see (44.9)] and volume [see (44.8)] charges is superposed onto the field set up by the free charges (let their density be ρ). Consequently,

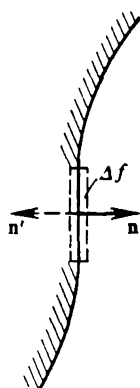


Fig. 44.2.

the potential at a point determined by the position vector \mathbf{r} is

$$\varphi(\mathbf{r}) = \int_{V_\rho} \frac{\rho(\mathbf{r}') dV'}{R} + \int_{f_d} \frac{P_n(\mathbf{r}') df'}{R} - \int_{V_d} \frac{\nabla' P(\mathbf{r}') dV'}{R} \quad (44.10)$$

where for brevity we have used the notation $R = |\mathbf{r} - \mathbf{r}'|$; dV' is an elementary volume taken in the vicinity of the point \mathbf{r}' , and df' is an element of the surface of the dielectric taken in the vicinity of the point \mathbf{r}' ; the divergence of $\mathbf{P}(\mathbf{r}')$ is taken over the primed coordinates, therefore the operator ∇ is primed [see (XI.52)]. The first integral is taken over the volume where ρ is non-zero, the second integral is taken over all the surfaces confining the dielectric, and, finally, the third integral is taken over the entire volume of the dielectric.

45. Description of the Field in Dielectrics

If ∇P differs from zero, every elementary volume of a dielectric is equivalent to a point charge of the magnitude $-\nabla P dV$ and makes the corresponding contribution to the macroscopic field \mathbf{E} . Therefore, when dielectrics are present, Eq. (42.3) must be written as

$$\nabla \mathbf{E} = 4\pi(\rho_{\text{free}} + \rho_{\text{bound}}) = 4\pi(\rho - \nabla P) \quad (45.1)$$

(by ρ is meant the density of the free charges). The equation in this form is hardly suitable for finding \mathbf{E} because it determines the latter not only in terms of the density of the free charges, but also through the nature of polarization of the dielectric. The polarization, in turn, is a function of \mathbf{E} .

It is easy to note that if we introduce the auxiliary quantity

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \quad (45.2)$$

the following equation holds for it:

$$\nabla \mathbf{D} = 4\pi\rho \quad (45.3)$$

that is, $\nabla \mathbf{D}$ is determined only by the density of the free charges.

The quantity \mathbf{D} is called the **electric displacement** (or the **electric induction**).

A comparison of Eqs. (45.1) and (45.3) shows that the operation of finding \mathbf{D} is much simpler than that of finding \mathbf{E} . There would be little good from the quantity \mathbf{D} , however, if it were not for the circumstance that in the majority of practically important cases \mathbf{D} is proportional to \mathbf{E} . It is therefore possible to use a "roundabout manoeuvre": instead of the main characteristic of a field \mathbf{E} , first the auxiliary quantity \mathbf{D} is calculated, and then the transition is performed from \mathbf{D} to \mathbf{E} .

The introduction of the quantity \mathbf{D} is also expedient because many formulas written with the use of \mathbf{D} are much simpler than they would be when expressed in terms of \mathbf{E} and \mathbf{P} .

Experiments show that in many cases having a practical interest, the polarization is proportional to the field strength:

$$\mathbf{P} = \chi \mathbf{E} \quad (45.4)$$

(we are meanwhile limiting ourselves to a treatment of isotropic dielectrics). The quantity χ is known as the **electric susceptibility** of a substance. It is always positive. Introducing (45.4) into (45.2), we find that

$$\mathbf{D} = \mathbf{E} + 4\pi\chi\mathbf{E} = \epsilon\mathbf{E} \quad (45.5)$$

where

$$\epsilon = 1 + 4\pi\chi \quad (45.6)$$

is the **relative permittivity** or simply the **permittivity**¹ of a substance (also sometimes called the **dielectric constant**).

Hence, \mathbf{D} and \mathbf{E} are often proportional to each other. This is exactly why it is expedient to introduce \mathbf{D} .

Proceeding from Eq. (45.3), it would seem possible for us to conclude that \mathbf{D} is determined only by the density of the free charges. But this is not true. Equation (45.3) alone is not sufficient for determining \mathbf{D} . To understand this more clearly, let us recall how \mathbf{E} is found in the absence of a dielectric. In addition to the equation $\nabla \cdot \mathbf{E} = 4\pi\rho$, we took advantage of the fact that $[\nabla \mathbf{E}] = 0$ and assumed that $\mathbf{E} = -\nabla\varphi$. Introducing this expression into the equation for the divergence, we arrive at Poisson's equation: $\nabla^2\varphi = -4\pi\rho$. Solution of this equation allows us to find φ , and then \mathbf{E} .

If we follow the same path in finding \mathbf{D} , in addition to (45.3) we must consider the equation

$$[\nabla \mathbf{D}] = [\nabla, \epsilon \mathbf{E}] = [\nabla \epsilon, \mathbf{E}] + \epsilon [\nabla \mathbf{E}] = [\nabla \epsilon, \mathbf{E}]$$

[we have used formula (XI.27) and taken into account that $[\nabla \mathbf{E}] = 0$]. This equation transforms into $[\nabla \mathbf{D}] = 0$ only when $\nabla \epsilon = 0$, i.e. when the dielectric is homogeneous. In the general case, $[\nabla \mathbf{D}]$ depends on \mathbf{E} , i.e. in the long run on the bound charges.

Although \mathbf{D} , however, depends in general on the bound charges, the collection of equations

$$[\nabla \mathbf{E}] = 0, \quad \nabla \cdot \mathbf{D} = 4\pi\rho, \quad \mathbf{D} = \epsilon \mathbf{E}$$

allows us to calculate \mathbf{E} and \mathbf{D} according to the known distribution of the free charges in space.

Using relation (45.4), we can determine the conditions in which ρ_{bound} differs from zero. Let us introduce the value given by (45.4)

¹ The absolute permittivity of a medium $\epsilon_a = \epsilon_0 \epsilon$ is introduced in electrical engineering. It is deprived of a physical meaning, however, and we shall not use it.

into formula (44.8):

$$\rho_{\text{bound}} = -\nabla P = -\nabla(\chi E) = -\chi \nabla E - E \nabla \chi \quad (45.7)$$

[see (XI.26)]. By (45.1), we have $\nabla E = 4\pi(\rho_{\text{free}} + \rho_{\text{bound}})$. Substituting this value for ∇E in formula (45.7), we get

$$\rho_{\text{bound}} = -\chi 4\pi(\rho_{\text{free}} + \rho_{\text{bound}}) - E \nabla \chi$$

Now let us solve the equation obtained relative to ρ_{bound} :

$$\rho_{\text{bound}} = -\frac{1}{\varepsilon} (4\pi\chi\rho_{\text{free}} + E \nabla \chi)$$

Inspection of the relation we have found shows that ρ_{bound} is non-zero, first, at the points where ρ_{free} is non-zero, and, second, at the points where $\nabla \chi \neq 0$, i.e. at places where the dielectric is non-homogeneous. We must note that space bound charges do not appear in a homogeneous polarized dielectric ($P = \text{const}$).

Let us consider the field in a homogeneous dielectric. Assume that in the absence of a dielectric with a given distribution of the free charges, a field is produced that is characterized by the strength E_0 and the potential φ_0 . We know E_0 and φ_0 to be the solutions of the equations

$$\nabla E_0 = 4\pi\rho \quad (45.8)$$

$$\Delta \varphi_0 = -4\pi\rho \quad (45.9)$$

[see (42.3) and (42.4)].

Now, without changing the arrangement of the free charges (i.e. ρ), let us fill the entire space in which the field is non-zero with a homogeneous ($\varepsilon = \text{const}$) isotropic dielectric. The field strength will therefore become equal to E , and the potential will be φ . Let us write equations for E and φ . According to (45.3)

$$\nabla D = \nabla(\varepsilon E) = 4\pi\rho \quad (45.10)$$

Substituting $-\nabla\varphi$ for E in this equation and taking into account that $\varepsilon = \text{const}$, we can write

$$-\nabla(\varepsilon \nabla \varphi) = -\nabla^2(\varepsilon \varphi) = 4\pi\rho$$

or

$$\Delta(\varepsilon \varphi) = -4\pi\rho \quad (45.11)$$

A comparison of (45.10) with (45.8) and (45.11) with (45.9) shows that the equation for εE coincides with that for E_0 , and the equation for $\varepsilon \varphi$ with that for φ_0 . It follows that the filling of a space in which a field is present with a homogeneous isotropic dielectric leads to both the field strength and the potential becoming equal to $1/\varepsilon$ -th of their initial values. Particularly, for the field of a point charge placed in a homogeneous dielectric, we have

$$E = \frac{e r}{\varepsilon r^3}, \quad \varphi = \frac{e}{\varepsilon r} \quad (45.12)$$

It must be noted that by (45.11), Poisson's equation (42.4) for a field in a homogeneous isotropic dielectric is as follows:

$$\Delta\varphi = -\frac{4\pi}{\epsilon}\rho \quad (45.13)$$

We shall indicate without a derivation (which can be found in many textbooks of general physics) the conditions which \mathbf{E} and \mathbf{D} must satisfy at the interface between two dielectrics:

$$\left. \begin{aligned} E_{\tau 1} &= E_{\tau 2}, & \epsilon_1 E_{n1} &= \epsilon_2 E_{n2} \\ D_{n1} &= D_{n2}, & D_{\tau 1}/\epsilon_1 &= D_{\tau 2}/\epsilon_2 \end{aligned} \right\} \quad (45.14)$$

(the subscripts τ and n denote the tangential and normal components of the relevant vector, respectively).

It is a very difficult task to find the field by solving Eqs. (42.4), (45.3), etc. in the general case.

In cases involving symmetry, it is possible to establish the form of the field without solving any equations. We shall show this using the following example. Assume that we have the plane boundary of two semiinfinite homogeneous and isotropic dielectrics with the permittivities ϵ_1 and ϵ_2 . There is a point charge q in the first dielectric at the distance a from the boundary. We are to find the field in both dielectrics.

We shall form the field in the first dielectric from those of the point charge q and its mirror image—the imaginary charge q' . This assumption satisfies the main condition that the first dielectric contains only one source of \mathbf{D} —the charge q (q' is outside the first dielectric). We shall thus seek the potential in the first dielectric in the form

$$\varphi_1 = \frac{q}{\epsilon_1 r} + \frac{q'}{\epsilon_1 r'} \quad (45.15)$$

(what r and r' are is clear from Fig. 45.1).

We shall represent the field in the second dielectric as that of the imaginary charge q'' placed where q is. This assumption agrees with the fact that there are no sources of \mathbf{D} in the second dielectric (q'' is outside the second dielectric). We shall thus seek the potential in

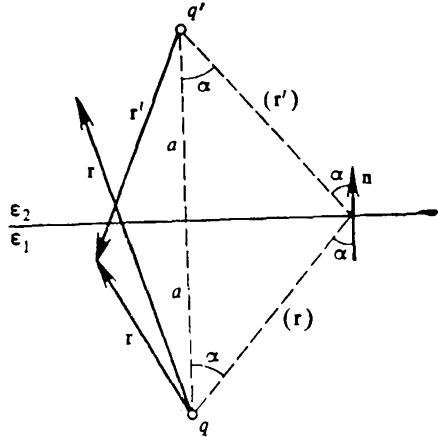


Fig. 45.1.

the second dielectric in the form

$$\varphi_2 = \frac{q''}{\varepsilon_2 r} \quad (45.16)$$

Let us try to choose the values of q' and q'' so as to satisfy the boundary conditions (45.14) for \mathbf{D} . According to our assumptions, \mathbf{D} has the following values in the first and second dielectrics:

$$\mathbf{D}_1 = \frac{q\mathbf{r}}{r^3} + \frac{q'\mathbf{r}'}{r'^3}, \quad \mathbf{D}_2 = \frac{q''\mathbf{r}}{r^3} \quad (45.17)$$

At the boundary of the dielectrics, $|\mathbf{r}| = |\mathbf{r}'|$. Consequently,

$$\mathbf{D}_{1b} = \frac{q\mathbf{r} + q'\mathbf{r}'}{r^3}, \quad \mathbf{D}_{2b} = \frac{q''\mathbf{r}}{r^3} \quad (45.18)$$

Let us find the normal components of the vectors in (45.18). We direct the vector \mathbf{n} from the first dielectric to the second one. Taking into account that the projection of \mathbf{r} onto \mathbf{n} equals a , and the projection of \mathbf{r}' equals $-a$, we get

$$D_{1n} = \frac{q - q'}{r^3} a, \quad D_{2n} = \frac{q''}{r^3} a$$

Since the tangential components of the vectors \mathbf{r} and \mathbf{r}' are the same, i.e. $r_\tau = r'_\tau$, we can write

$$D_{1\tau} = \frac{q + q'}{r^3} r_\tau, \quad D_{2\tau} = \frac{q''}{r^3} r_\tau \quad (45.19)$$

It follows from the equality $D_{1n} = D_{2n}$ that

$$q - q' = q'' \quad (45.20)$$

After introducing the values from (45.19) into the relation $D_{1\tau}/\varepsilon_1 = D_{2\tau}/\varepsilon_2$ [see (45.14)], we obtain

$$q + q' = \frac{\varepsilon_1}{\varepsilon_2} q'' \quad (45.21)$$

The simultaneous solution of Eqs. (45.20) and (45.21) yields the values of q' and q'' satisfying the boundary conditions:

$$q' = q \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}, \quad q'' = q \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \quad (45.22)$$

We have succeeded in "constructing" the functions \mathbf{D}_1 and \mathbf{D}_2 [see (45.17)], each of which satisfies the equation $\nabla \mathbf{D} = 4\pi\rho$ in its region. In addition, these functions satisfy the boundary conditions. Therefore, the functions (45.17) and, accordingly, (45.15) and (45.16) are the solutions of the problem [after introducing the values (45.22) for q' and q'' into them]. According to the uniqueness theorem (see Sec. 42), there are no other solutions.

46. Field in Anisotropic Dielectrics

In anisotropic dielectrics, the directions of the vectors \mathbf{P} and \mathbf{E} , generally speaking, do not coincide (see Appendix X). The relation between the components of these vectors is given by the expressions

$$P_i = \sum_{k=1}^3 \chi_{ik} E_k \quad (i = 1, 2, 3) \quad (46.1)$$

where χ_{ik} is a symmetric tensor of rank two known as the electrical susceptibility tensor.

By (45.2), we have $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ or, in components,

$$D_i = E_i + 4\pi P_i$$

Introducing P_i from (46.1), we get

$$D_i = E_i + 4\pi \sum_k \chi_{ik} E_k$$

If we represent E_i in the form $\sum_k \delta_{ik} E_k$, we can write

$$D_i = \sum_k \delta_{ik} E_k + 4\pi \sum_k \chi_{ik} E_k = \sum_k (\delta_{ik} + 4\pi\chi_{ik}) E_k$$

The quantities

$$\varepsilon_{ik} = \delta_{ik} + 4\pi\chi_{ik} \quad (46.2)$$

are clearly the components of a symmetric tensor of rank two. It is called the **permittivity tensor**. With its aid, the relation between the vectors \mathbf{D} and \mathbf{E} can be written as

$$D_i = \sum_k \varepsilon_{ik} E_k \quad (i = 1, 2, 3) \quad (46.3)$$

The symmetric tensor has six independent components. If it is reduced to the principal axes, it appears as follows:

$$(\varepsilon_{ik}) = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

We must note that since the principal values of the tensor χ_{ik} are positive, those of the tensor ε_{ik} are always greater than unity.

In crystals of the triclinic, monoclinic, and rhombic systems, all three principal values of the tensor ε_{ik} and, consequently, the semiaxes of the tensor ellipsoid are different. Such crystals are known as biaxial ones.

In crystals of the tetragonal, rhombohedral, and hexagonal systems, two principal values coincide: $\epsilon_1 = \epsilon_2 \neq \epsilon_3$. The tensor ellipsoid in this case is an ellipsoid of revolution. Such crystals are known as uniaxial ones.

In crystals of the cubic system, all three principal values of the tensor ϵ_{ik} are the same so that the tensor has the form $\epsilon\delta_{ik}$. The tensor ellipsoid in this case degenerates into a sphere. Such crystals do not differ from isotropic bodies in their dielectric (and optical) properties.

Chapter IX

MAGNETOSTATICS

47. Stationary Magnetic Field in a Vacuum

A point charge e moving at the velocity \mathbf{v} experiences in a magnetic field the force¹

$$\mathbf{F} = \frac{e}{c} [\mathbf{v}\mathbf{B}] \quad (47.1)$$

(c is the speed of light in a vacuum). The vector quantity \mathbf{B} called the **magnetic induction** is the basic (strength) characteristic of a magnetic field. Relation (47.1) may be considered as the definition of the quantity \mathbf{B} .

Owing to the absence in nature of magnetic charges similar to the electric charges² e , the lines of the vector \mathbf{B} have neither a beginning nor an end. This is why the flux of the vector \mathbf{B} through any closed surface always equals zero:

$$\oint_{\mathcal{f}} B_n df = 0 \quad (47.2)$$

Formula (47.2) is an analytical expression of Gauss's theorem for the magnetic induction vector.

Using the Ostrogradsky-Gauss theorem, expression (47.2) can be written as

$$\int_V \nabla \mathbf{B} dV = 0 \quad (47.3)$$

Condition (47.3) must be observed for any arbitrarily chosen volume V . This is possible only if the integrand function is zero at each point. We thus arrive at the conclusion that the divergence of the magnetic induction vector is zero at every point of a field:

$$\nabla \mathbf{B} = 0 \quad (47.4)$$

¹ The sum of the forces (41.1) and (47.1) is called the Lorentz force.

² Proceeding from the fact that the equations of physics in general and those of electrodynamics in particular must be symmetric, Paul Dirac advanced the hypothesis that magnetic charges (Dirac's monopoles) ought to exist in nature. Searches for these charges have meanwhile given no results, so that the question of the existence of Dirac's monopoles remains open.

It is proved in vector analysis [see (XI.44)] that the divergence of the curl of a vector function always equals zero. Therefore, the magnetic induction can be represented as the curl of a certain vector function \mathbf{A} :

$$\mathbf{B} = [\nabla \mathbf{A}] \quad (47.5)$$

The function \mathbf{A} is called the **vector potential** of a magnetic field.

The vector potential, like its scalar counterpart φ , is determined non-uniquely. Indeed, since the curl of the gradient of any function is zero [see (XI.43)], the addition to the vector potential of the quantity $\nabla\psi$ (here ψ is an arbitrary function) does not change the value of $[\nabla \mathbf{A}]$, i.e. \mathbf{B} . Hence, if \mathbf{A} is the vector potential corresponding to a given magnetic field, the function

$$\mathbf{A}' = \mathbf{A} + \nabla\psi \quad (47.6)$$

is also a vector potential of this field.

The property (47.6) allows us to choose the potential in the most convenient way, for instance, to impose definite conditions on the divergence of \mathbf{A} . Indeed, it can be seen from (47.6) that

$$\nabla \mathbf{A}' = \nabla \mathbf{A} + \nabla (\nabla\psi) = \nabla \mathbf{A} + \Delta\psi$$

so that the proper choice of ψ can give $\nabla \mathbf{A}'$ any preset value. Within the scope of magnetostatics, we shall choose ψ so that

$$\nabla \mathbf{A} = 0 \quad (47.7)$$

To illustrate what has been said above, let us consider the vector potential of the homogeneous magnetic field $\mathbf{B} = \text{const} = \mathbf{B}_0$. Let us direct the z -axis along \mathbf{B}_0 . Hence $B_x = B_y = 0$, $B_z = B_0$, and Eq. (47.5) written in components becomes

$$\begin{aligned} B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0, & B_y &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0, \\ B_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0 \end{aligned} \quad (47.8)$$

It can be seen that these equations are satisfied, say, by the following value of the potential:

$$A_x = -B_0 y, \quad A_y = 0, \quad A_z = 0 \quad (47.9)$$

Figure 47.1a depicts the lines of the vector \mathbf{A} having the components (47.9).

Solution (47.9) is not the only one. Equations (47.8) are also satisfied by the following potential:

$$A_x = 0, \quad A_y = B_0 x, \quad A_z = 0 \quad (47.10)$$

The lines of \mathbf{A} for this case are shown in Fig. 47.1b. It is evident that

$$A_x = -\alpha B_0 y, \quad A_y = (1 - \alpha) B_0 x, \quad A_z = 0$$

where α is any number, will also be a solution. Particularly, Eqs. (47.8) are satisfied by

$$A_x = -\frac{1}{2} B_0 y, \quad A_y = \frac{1}{2} B_0 x, \quad A_z = 0 \quad (47.11)$$

Solution (47.11) can be represented as

$$A_x = \frac{1}{2} (B_y z - B_z y), \quad A_y = \frac{1}{2} (B_z x - B_x z), \quad A_z = \frac{1}{2} (B_x y - B_y x)$$

(remember that $B_x = B_y = 0$), whence

$$\mathbf{A} = \frac{1}{2} [\mathbf{B}_0 \mathbf{r}] = \frac{1}{2} [\mathbf{B}_0 \mathbf{R}] \quad (47.12)$$

where \mathbf{R} is the component of the position vector \mathbf{r} perpendicular to the z -axis. The lines of the vector \mathbf{A} corresponding to (47.12) are shown in Fig. 47.1c.

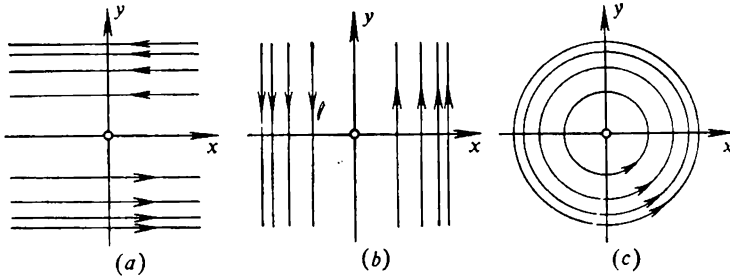


Fig. 47.1.

All the values of the potential we have given satisfy condition (47.7). We thus conclude that Eqs. (47.5) and (47.7) do not completely determine \mathbf{A} . For its determination to be unique, we must set the boundary conditions for it.

48. Poisson's Equation for the Vector Potential

It is known from the general course of physics that the circulation of the vector \mathbf{B} around any closed contour Γ taken in a stationary magnetic field (in a field of steady currents) is proportional to the algebraic sum of the currents enclosed by the contour:

$$\oint_{\Gamma} B_l dl = \frac{4\pi}{c} \sum i \quad (48.1)$$

We shall treat formula (48.1) as a relation established experimentally.

Introducing the current density \mathbf{j} , we can represent the sum of the currents as the flux of the vector \mathbf{j} through the surface confined by the contour Γ . Formula (48.1) thus becomes

$$\oint_{\Gamma} B_l dl = \frac{4\pi}{c} \int_f \mathbf{j} d\mathbf{f}$$

Let us transform the left-hand side of the relation we have obtained according to Stokes' theorem. The result is

$$\int_f [\nabla \mathbf{B}] d\mathbf{f} = \frac{4\pi}{c} \int_f \mathbf{j} d\mathbf{f} \quad (48.2)$$

Assume that integration on the left-hand and right-hand sides is performed over the same surface [although Eq. (48.2) is also observed for different surfaces if only they rest on the same contour Γ]. Relation (48.2) must be observed for any arbitrarily taken surface f . This is only possible if at every point of the field

$$[\nabla \mathbf{B}] = \frac{4\pi}{c} \mathbf{j} \quad (48.3)$$

(we remind our reader that we are considering the field of steady currents).

Equation (48.3) plays the same basic role in magnetostatics as Eq. (42.3) does in electrostatics. Together with Eq. (47.4), it allows us to calculate the field of preset stationary currents.

Let us introduce the curl of \mathbf{A} [see (47.5)] instead of \mathbf{B} into formula (48.3):

$$[\nabla, [\nabla \mathbf{A}]] = \frac{4\pi}{c} \mathbf{j}$$

According to (XI.45), we have $[\nabla, [\nabla \mathbf{A}]] = \nabla(\nabla \mathbf{A}) - \Delta \mathbf{A}$. Assuming, as we agreed upon [see (47.7)], that $\nabla \mathbf{A} = 0$, we get the following differential equation for \mathbf{A} :

$$\Delta \mathbf{A} = -\frac{4\pi}{c} \mathbf{j} \quad (48.4)$$

This vector equation is equivalent to three scalar equations:

$$\Delta A_k = -\frac{4\pi}{c} j_k \quad (k = x, y, z) \quad (48.5)$$

each of which is similar to Poisson's equation for φ [see (42.4)]. Equations (42.4) and (48.5) are equivalent from the mathematical viewpoint. Consequently, substituting A_k for φ and j_k/c for ρ in the solution of Eq. (42.4), we get the solution of Eq. (48.5). With a view to formula (42.7), we obtain

$$A_k = \frac{1}{c} \int \frac{j_k(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|} \quad (k = x, y, z) \quad (48.6)$$

where the integral is evaluated over the entire region in which the currents producing the field flow.

The three expressions (48.6) can be combined into a single vector one:

$$\mathbf{A} = \frac{1}{c} \int \frac{\mathbf{j}(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|} \quad (48.7)$$

Formula (48.7) allows us to calculate the vector potential of the field set up by currents according to their known distribution in space. Let us consider the field of a line (straight) current as an example. It is general knowledge that the potential of the electric field produced by a thin infinitely long uniformly charged filament can be written as¹ (we are considering the field outside the filament)

$$\varphi(R) = -2\lambda \ln \frac{R}{R_0} = -2\rho\sigma \ln \frac{R}{R_0} \quad (48.8)$$

where R is the distance from the filament, R_0 is the distance to points whose potential is taken as zero (in the given case we cannot assume that $\varphi = 0$ at infinity because with such normalization the potential is infinitely great at finite values of R), and λ is the linear density of the charge which, assuming the latter to be uniformly distributed over the entire cross section of the filament, can be written as $\rho\sigma$ (here σ is the cross-sectional area of the filament).

Now assume that we have a thin infinitely long straight wire through which a current of density j flows that is uniformly distributed over the cross section σ . Directing the z -axis along the wire, we have $j_x = j_y = 0$, $j_z = j$. On the grounds that Eqs. (42.4) and (48.5) are identical, we can obtain an expression for A_z by substituting $j_z/c = j/c$ for ρ in (48.8):

$$A_z(R) = -\frac{2j\sigma}{c} \ln \frac{R}{R_0} = -\frac{2i}{c} \ln \frac{R}{R_0} \quad (48.9)$$

where $i = j\sigma$ is the current flowing through the wire. The introduction into (48.8) of $j_x = 0$ and $j_y = 0$ instead of ρ yields zero values for the components A_x and A_y . Hence, the vector \mathbf{A} can be written as

$$\mathbf{A} = -\frac{2i}{c} \ln \frac{R}{R_0} \mathbf{e}_z \quad (48.10)$$

where \mathbf{e}_z is the unit vector of the z -axis.

Taking the curl of expression (48.10), we find \mathbf{B} :

$$\begin{aligned} \mathbf{B} = [\nabla \mathbf{A}] &= -\frac{2i}{c} \left[\nabla, \left(\ln \frac{R}{R_0} \mathbf{e}_z \right) \right] \\ &= -\frac{2i}{c} \left[\left(\nabla \ln \frac{R}{R_0} \right), \mathbf{e}_z \right] - \frac{2i}{c} \ln \frac{R}{R_0} [\nabla \mathbf{e}_z] \end{aligned}$$

¹ It is a simple matter to find that $E(R) = 2\lambda/R$ with the aid of Gauss's theorem. Consequently, $d\varphi/dR = -2\lambda/R$. Integration leads to formula (48.8).

[we have used formula (XI.27)]. Since $\mathbf{e}_z = \text{const}$, the second term vanishes. The gradient of the function $\ln(R/R_0)$ equals \mathbf{R}/R^2 . Therefore,

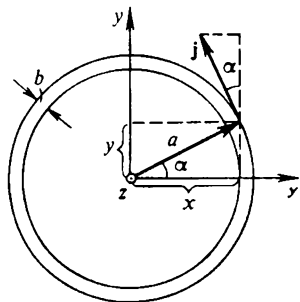
$$\mathbf{B} = -\frac{2i}{c} \left[\frac{\mathbf{R}}{R^2}, \mathbf{e}_z \right] = \frac{2i}{cR^2} [\mathbf{e}_z, \mathbf{R}] \quad (48.11)$$

According to the result we have obtained, the vector \mathbf{B} at every point is in a plane perpendicular to the wire and directed along a tangent to the circle encompassing the wire. The magnitude of \mathbf{B} is

$$B = \frac{2i}{cR}$$

49. Field of Solenoid

Let us calculate the field of an infinite solenoid, representing it as an infinite cylinder of radius a in whose surface layer of thickness b (where $b \ll a$) a current of density j flows (here bj is equivalent to ni , where i is the current in the solenoid, and n is the number of its turns per unit length).



We choose rectangular coordinate axes and make the z -axis coincide with the geometrical axis of the solenoid (Fig. 49.1). The projections of the vector \mathbf{j} onto the coordinate axes are

$$j_x = -j \sin \alpha = -j \frac{y}{a},$$

$$j_y = j \cos \alpha = j \frac{x}{a}, \quad j_z = 0 \quad (49.1)$$

Fig. 49.1.

It immediately follows from $j_z = 0$ that $A_z = 0$. In accordance with what was said in Sec. 48, the component A_x coincides with the potential φ produced by the charge distributed in the surface layer of the cylinder with the density $\rho_x = j_x/c = -(j/c)(y/a) = -\rho_0(y/a)$. We have introduced the symbol

$$\rho_0 = \frac{j}{c} = \frac{ni}{bc} \quad (49.2)$$

Similarly, the component A_y coincides with the potential of the charge distributed in the surface layer with the density $\rho_y = j_y/c = \rho_0(x/a)$.

The density, which changes according to the law $-y/a$, can be obtained as follows. Let us insert a cylinder charged uniformly with the volume density $+\rho_0$ into another cylinder charged uniformly

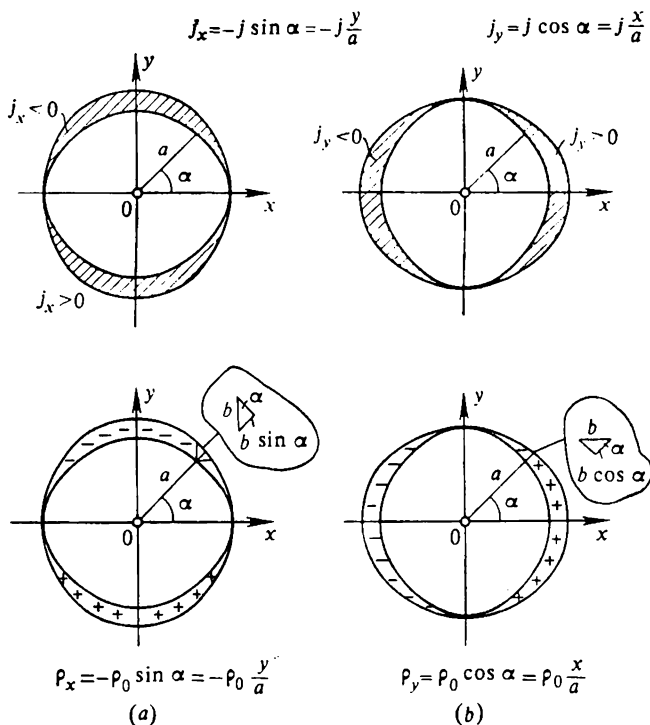


Fig. 49.2.

with the volume density $-\rho_0$. As a whole, the system of two cylinders is neutral at every point. Now let us move the negatively charged cylinder over the length $b/2$ in the direction of the y -axis, and the positively charged cylinder over the same distance in the opposite direction (Fig. 49.2a). Since $b \ll a$ (here a is the radius of the cylinders), the formed system may be considered as a cylinder in whose surface layer of thickness b a charge having a varying density ρ_x is distributed. Actually, the density of the charge in the surface layer is constant and equals ρ_0 (or $-\rho_0$), while what does change is the thickness of the uncompensated layer. A glance at Fig. 49.2a shows that this thickness varies according to the law $b \sin \alpha$ so that the charge per unit of cylinder surface is $-\rho_0 b \sin \alpha = -\rho_0 b y/a$. Our error will not be noticeable, however, if we consider the thickness of the charged layer to be the same everywhere and equal to b , and the charge density in this layer to vary according to the law

$$\rho_x = -\frac{\rho_0 y}{a} \quad (49.3)$$

To obtain a density varying according to the law x/a , let us shift the cylinders as shown in Fig. 49.2*b*. Now the density of the charge in an imaginary surface layer of thickness b will vary according to the law

$$\rho_y = \frac{\rho_0 x}{a} \quad (49.4)$$

In the cases being considered, the fields are the superposition of the two fields $+\varphi_0$ and $-\varphi_0$, identical in magnitude but opposite in sign, displaced relative to each other over the very small distance \mathbf{b} (the latter is directed oppositely to the y -axis for ρ_x , and along the x -axis for ρ_y). Assume that initially the fields were accurately superposed, as a result of which the potential at the point determined by the position vector \mathbf{r} was $[\varphi_0(\mathbf{r})] + [-\varphi_0(\mathbf{r})]$, i.e. zero. Now let us shift the field $+\varphi_0$ by $\mathbf{b}/2$, and the field $-\varphi_0$ by $-\mathbf{b}/2$. Hence, the potential $+\varphi_0$ that was at the point $(\mathbf{r} - \mathbf{b}/2)$ and the potential $-\varphi_0$ that was at the point $(\mathbf{r} + \mathbf{b}/2)$ will be at the point with the position vector \mathbf{r} . Consequently,

$$\varphi(\mathbf{r}) = \varphi_0\left(\mathbf{r} - \frac{\mathbf{b}}{2}\right) - \varphi_0\left(\mathbf{r} + \frac{\mathbf{b}}{2}\right)$$

Owing to the smallness of \mathbf{b} , both terms of the expression we have obtained can be transformed by formula (XI.5), i.e. can be written as

$$\begin{aligned} \varphi(\mathbf{r}) &= [\varphi_0(\mathbf{r}) - \nabla\varphi_0 \cdot \mathbf{b}/2] - [\varphi_0(\mathbf{r}) + \nabla\varphi_0 \cdot \mathbf{b}/2] \\ &= -\nabla\varphi_0 \cdot \mathbf{b} = -\left(\frac{\partial\varphi_0}{\partial x} b_x + \frac{\partial\varphi_0}{\partial y} b_y + \frac{\partial\varphi_0}{\partial z} b_z\right) \end{aligned}$$

For ρ_x , the vector \mathbf{b} has the components $b_x = b_z = 0$, and $b_y = -b$, so that

$$\varphi(\mathbf{r}) = \frac{\partial\varphi_0}{\partial y} b \quad (\text{for } A_x) \quad (49.5)$$

for ρ_y , we have $b_y = b_z = 0$, and $b_x = b$, so that

$$\varphi(\mathbf{r}) = -\frac{\partial\varphi_0}{\partial x} b \quad (\text{for } A_y) \quad (49.6)$$

It now remains to find φ_0 and introduce its derivatives into (49.5) and (49.6). Recall that φ_0 is the potential of the field produced by a cylinder of radius a charged with the constant volume density $+\rho_0$. The potential inside and outside such a cylinder is determined by different formulas.

Field Inside a Solenoid. We can easily find with the aid of Gauss's theorem that the field inside a charged cylinder is $E = 2\pi\rho_0 R$, where R is the distance from the axis of the cylinder ($R < a$). The potential $\varphi_0 = -\pi\rho_0 R^2 + \text{const} = -\pi\rho_0 (x^2 + y^2) + \text{const}$ corre-

sponds to this field. Its derivatives are

$$\frac{\partial \varphi_0}{\partial x} = -2\pi\rho_0 x, \quad \frac{\partial \varphi_0}{\partial y} = -2\pi\rho_0 y$$

Let us introduce the found values of the derivatives into (49.5) and (49.6):

$$\begin{aligned} \varphi(\mathbf{r}) &= -2\pi\rho_0 by & (\text{for } A_x) \\ \varphi(\mathbf{r}) &= 2\pi\rho_0 bx & (\text{for } A_y) \end{aligned}$$

Substituting ni/bc for ρ_0 in these expressions [see (49.2)], we obtain expressions for A_x and A_y (it was established above that $A_z = 0$):

$$\left. \begin{aligned} A_x &= -\frac{2\pi ni}{c} y = -\frac{1}{2} \left(\frac{4\pi ni}{c} \right) y \\ A_y &= \frac{2\pi ni}{c} x = \frac{1}{2} \left(\frac{4\pi ni}{c} \right) x \\ A_z &= 0 \end{aligned} \right\} \quad (49.7)$$

It was shown in Sec. 47 [see (47.11)] that the potential (49.7) determines a homogeneous magnetic field with the magnetic induction

$$B = \frac{4\pi}{c} ni \quad (49.8)$$

parallel to the z -axis, i.e. to the axis of the solenoid.

Field Outside a Solenoid. The potential outside a cylinder charged with the density ρ_0 is

$$\varphi_0 = -2\rho_0\pi a^2 \ln \frac{R}{R_0}$$

where R is the distance from the cylinder axis ($R > a$), and R_0 is a constant [compare with (48.8)]. The derivatives of φ_0 can be written as

$$\frac{\partial \varphi_0}{\partial x} = -2\rho_0\pi a^2 \frac{x}{R^2}, \quad \frac{\partial \varphi_0}{\partial y} = -2\rho_0\pi a^2 \frac{y}{R^2}$$

Substitution of these values into (49.5) and (49.6) yields

$$\begin{aligned} \varphi(\mathbf{r}) &= -2\rho_0\pi a^2 b \frac{y}{R^2} & (\text{for } A_x) \\ \varphi(\mathbf{r}) &= 2\rho_0\pi a^2 b \frac{x}{R^2} & (\text{for } A_y) \end{aligned}$$

Substituting ni/bc for ρ_0 , we obtain A_x and A_y . Let us write the values of all three components:

$$\left. \begin{aligned} A_x &= -\frac{2\pi nia^2}{c} \frac{y}{R^2} = -K \frac{y}{R^2} \\ A_y &= \frac{2\pi nia^2}{c} \frac{x}{R^2} = K \frac{x}{R^2} \\ A_z &= 0 \end{aligned} \right\} \quad (49.9)$$

The quantity $R^2 = x^2 + y^2$ does not contain z . Consequently, $\partial A_x / \partial z = \partial A_y / \partial z = 0$. Since $A_z = 0$, it follows that $\partial A_z / \partial x = \partial A_z / \partial y = 0$. Hence, B_x and B_y vanish. Let us find B_z :

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = K \left[\left(\frac{1}{R^2} - \frac{2x}{R^3} \frac{x}{R} \right) - \left(-\frac{1}{R^2} + \frac{2y}{R^3} \frac{y}{R} \right) \right] \\ = K \left[\frac{2}{R^2} - \frac{2}{R^2} \frac{x^2 + y^2}{R^2} \right] = 0$$

We have thus found that the field outside an infinitely long solenoid is zero. The vector potential outside the solenoid, however, is non-zero. The collection of formulas (49.9) can be written as a single vector formula:

$$\mathbf{A} = \frac{2\pi n i a^2}{c R^2} [\mathbf{e}_z, \mathbf{R}] \quad (49.10)$$

A comparison with formula (48.11) shows that the field of the vector \mathbf{A} outside the solenoid has the same nature as the field of the vector \mathbf{B} around a straight long current-carrying conductor.

50. The Biot-Savart Law

The expression

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int \frac{\mathbf{j}(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|} \quad (50.1)$$

makes it possible, knowing the distribution of currents in space, to determine the vector potential of a magnetic field [see (48.7)]. Let us attempt to find a formula that would allow us to find \mathbf{B} directly according to the preset currents. For this purpose, let us calculate the curl of the function (50.1). It must be borne in mind that integration in (50.1) is performed over the primed coordinates x', y', z' , whereas differentiation in taking the curl is performed over the unprimed coordinates x, y, z . Therefore, we may exchange the places of the operations of integration and taking the curl. With this in view, we obtain

$$\mathbf{B}(\mathbf{r}) = [\nabla, \mathbf{A}(\mathbf{r})] = \frac{1}{c} \int \left[\nabla, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] dV'$$

Considering $\mathbf{j}(\mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ as the product of the vector function $\mathbf{j}(\mathbf{r}')$ and the scalar function $1/|\mathbf{r} - \mathbf{r}'|$, we shall use formula (XI.27):

$$\left[\nabla, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] = \left[\left(\nabla, \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right), \mathbf{j}(\mathbf{r}') \right] + \frac{1}{|\mathbf{r} - \mathbf{r}'|} [\nabla, \mathbf{j}(\mathbf{r}')]$$

The second term vanishes because $\mathbf{j}(\mathbf{r}')$ contains no unprimed coordinates. The gradient in the first term is

$$\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \nabla \frac{1}{\sqrt{\sum (x_i - x'_i)^2}} = - \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

We have thus arrived at the formula

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \frac{1}{c} \int \left[-\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \mathbf{j}(\mathbf{r}') \right] dV' \\ &= \frac{1}{c} \int \frac{[\mathbf{j}(\mathbf{r}'), (\mathbf{r}-\mathbf{r}')] dV'}{|\mathbf{r}-\mathbf{r}'|^3} \quad (50.2) \end{aligned}$$

(we have put the scalar factor outside the sign of the vector product and, in addition, have changed the places of the factors, which resulted in vanishing of the minus sign).

The formula obtained is a solution of the problem we posed—it allows us to calculate \mathbf{B} from the preset currents. Formula (50.2) is simplified if the currents flow only through thin wires. Figure 50.1

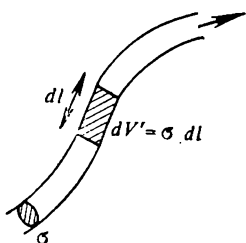


Fig. 50.1.

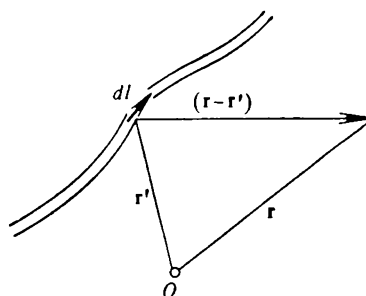


Fig. 50.2.

shows a portion of such a wire. Inspection of the figure shows that the expression $\mathbf{j} dV'$ can be written as

$$\mathbf{j} dV' = \mathbf{j} \sigma dl = \mathbf{j} \sigma dl = i dl \quad (50.3)$$

where σ is the cross-sectional area of the wire, i is the current, and dl is a vector coinciding with the wire element dl and having the same direction as the current. Substituting $i dl$ for $\mathbf{j} dV'$ in (50.2), we obtain

$$\mathbf{B} = \frac{1}{c} \int \frac{i [dl, (\mathbf{r}-\mathbf{r}')] }{|\mathbf{r}-\mathbf{r}'|^3} \quad (50.4)$$

(integration is performed over the length of all the wires).

Formula (50.4) is an analytical expression of the Biot-Savart law. Figure 50.2 explains that $\mathbf{r}-\mathbf{r}'$ is a vector drawn from the point where the current element dl is to the point for which \mathbf{B} is being calculated.

51. Magnetic Moment

Before turning to the topic of the present section, let us obtain the continuity equation, which is a corollary of the law of charge conservation.

Assume that currents characterized by the density $\mathbf{j} = \mathbf{j}(\mathbf{r})$ flow in a certain region. Let us separate an imaginary volume V confined by the surface f in this region. The charge flowing outward through this surface in unit time can be written as

$$\oint_f j_n df = \int_V \nabla \mathbf{j} dV$$

The above expression equals the rate of diminishing of the charge confined in the volume V , which is determined by the expression

$$-\frac{d}{dt} \int_V \rho dV = - \int_V \frac{\partial \rho}{\partial t} dV$$

(we have written the partial derivative with respect to t because ρ is a function not only of time, but also of the coordinates).

Equating the two expressions, we obtain

$$\int_V \nabla \mathbf{j} dV = - \int_V \frac{\partial \rho}{\partial t} dV$$

The above equation must be observed for any arbitrarily chosen volume. This is possible only upon equality of the integrands at every point of space. We thus arrive at the relation

$$\nabla \mathbf{j} = - \frac{\partial \rho}{\partial t} \quad (51.1)$$

known as the continuity equation in the differential form. In the integral form, the continuity equation is

$$\oint_f j_n df = - \frac{d}{dt} \int_V \rho dV \quad (51.2)$$

Now let us consider a system of stationary currents circulating within a restricted volume V and calculate the magnetic field set up by this system at distances that are great in comparison with the dimensions of the system. This problem is similar to that treated in Sec. 43.

First of all, we must note that owing to the stationary nature of the currents the accumulation or dissipation of charges cannot occur at any of the system's points, i.e. the condition $\partial \rho / \partial t = 0$ must hold everywhere. It thus follows by (51.1) that within the system

$$\nabla \mathbf{j} = 0 \quad (51.3)$$

Further, there are no currents outside the system. Hence, everywhere on the surface confining the system, we have

$$j_n = 0 \quad (51.4)$$

(the currents do not intersect the surface).

Finally, we shall prove that

$$\int_V \mathbf{j} dV = 0 \quad (51.5)$$

where the integral is evaluated over the entire volume of the system. Owing to the stationary and restricted nature of the system, all the current tubes¹ that can be separated inside the system are closed. Consequently, the entire volume of the system can be divided into closed current tubes. For each of the tubes, the integral over its volume can be transformed as follows:

$$\int_{\text{over tube}} \mathbf{j} dV = \int \mathbf{j} \sigma dl = \oint i dl = i \oint dl = 0 \quad (51.6)$$

[σ is the cross-sectional area of a tube, i is the current, and dl is an element of length of a tube, see formula (50.3)]. Summation of expression (51.6) over all the current tubes yields formula (51.5).

Let us choose the origin of coordinates inside the system and write an expression for the vector potential:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int_V \frac{\mathbf{j}(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|} \quad (51.7)$$

Here \mathbf{r} is the position vector of the point for which \mathbf{A} is being calculated, and \mathbf{r}' is the position vector of the point in whose vicinity the elementary volume dV' is; integration is performed over the primed coordinates within the confines of the system's volume.

Taking advantage of the fact that according to our assumption $\mathbf{r}' \ll \mathbf{r}$, let us expand expression (51.7) into a series² in \mathbf{r}'/\mathbf{r} . To within first-order terms, we have

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int_V \frac{\mathbf{j}(\mathbf{r}') dV'}{r} - \frac{1}{c} \int_V \mathbf{j}(\mathbf{r}') \left\{ \nabla \frac{1}{r} \mathbf{r}' \right\} dV' \quad (51.8)$$

¹ By a tube of an electric current is meant the same as by a tube of a flow in a fluid, i.e. the volume confined by lines tangents to which at each point coincide with the direction of the vector \mathbf{j} .

² Considering $-\mathbf{r}'$ as a small increment of the position vector \mathbf{r} , we write the function $f(\mathbf{r}) = 1/r$ for the value of the argument $\mathbf{r} - \mathbf{r}'$ as

$$f(\mathbf{r} - \mathbf{r}') = f(\mathbf{r}) + \nabla f(\mathbf{r}) \cdot (-\mathbf{r}')$$

The first term of (51.8), owing to (51.6), vanishes. Consequently, substituting for $\nabla (1/r)$ its value, we obtain

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int_V \frac{\mathbf{j}(\mathbf{r}') (\mathbf{r}\mathbf{r}') dV'}{r^3}$$

Let us write the k -th component of the potential:

$$A_k = \frac{1}{cr^3} \int j_k(\mathbf{r}') \left(\sum_i x_i x'_i \right) dV' = \frac{1}{cr^3} \sum_i x_i \int j_k x'_i dV' \quad (51.9)$$

(we have expressed the scalar product $\mathbf{r}\mathbf{r}'$ through the components of the position vectors). The products $j_k x'_i$ are the components of a tensor of rank two¹. Let us write this tensor as the sum of a symmetric and antisymmetric tensors:

$$j_k x'_i = \frac{j_k x'_i + j_i x'_k}{2} + \frac{j_k x'_i - j_i x'_k}{2} = S_{ki} + A_{ki} \quad (51.10)$$

[see (X.27)].

Let us prove that the integral of the symmetric component of the tensor (51.10) is zero. For this purpose, we shall use the identity

$$\nabla' (x'_i x'_k) \equiv \mathbf{j} \nabla' (x'_i x'_k) + x'_i x'_k \nabla' \mathbf{j} \quad (51.11)$$

[see (XI.26)]. In view of (51.3), the second term on the right-hand side vanishes. Let us write the expression $\nabla' (x'_i x'_k)$ using the formula for the gradient of the product of scalar functions [see (XI.25)]:

$$\begin{aligned} \nabla' (x'_i x'_k) &= x'_i \nabla' x'_k + x'_k \nabla' x'_i \\ &= x'_i \sum_m \mathbf{e}_m \frac{\partial x'_k}{\partial x'_m} + x'_k \sum_m \mathbf{e}_m \frac{\partial x'_i}{\partial x'_m} = \sum_m \mathbf{e}_m (x'_i \delta_{km} + x'_k \delta_{im}) \end{aligned}$$

(we have taken into account that $\partial x'_k / \partial x'_m = \delta_{km}$). The expression in parentheses is the m -th component of the gradient of the function $x'_i x'_k$.

Now let us calculate the second term of expression (51.11):

$$\begin{aligned} \mathbf{j} \nabla' (x'_i x'_k) &= \sum_m j_m \{ \nabla' (x'_i x'_k) \}_m = \sum_m j_m (x'_i \delta_{km} + x'_k \delta_{im}) \\ &= \sum_m j_m x'_i \delta_{km} + \sum_m j_m x'_k \delta_{im} \end{aligned}$$

In the first sum on the right-hand side, only the addend with $m = k$ is non-zero, and in the second—the addend with $m = i$. Hence

$$\mathbf{j} \nabla' (x'_i x'_k) = j_k x'_i + j_i x'_k$$

¹ The integral $I_{ki} = \int j_k x'_i dV'$ is also a component of a tensor. Consequently, the vector \mathbf{A} to within the factor $1/cr^3$ is the product of the tensor I_{ki} and the vector x_i [see (X.22)].

Therefore, by (51.11)

$$S_{ki} = \frac{j_h x'_i + j_i x'_h}{2} = \frac{1}{2} \nabla' (x'_i x'_h j)$$

Let us take an integral of this expression:

$$\int_V S_{ki} dV' = \frac{1}{2} \int_V \nabla' (x'_i x'_h j) dV' = \frac{1}{2} \oint_{\partial V} x'_i x'_h j_n df$$

(we have employed the Ostrogradsky-Gauss theorem). In view of (51.4), the last integral vanishes. We have thus proved that the integral of the symmetric component of the integrand on the right-hand side in (51.9) vanishes. This allows us to write expression (51.9) as

$$A_k = \frac{1}{cr^3} \sum_i x_i \int_V \frac{j_h x'_i - j_i x'_h}{2} dV' = \frac{1}{2cr^3} \int_V \left\{ \sum_i x_i (j_h x'_i - j_i x'_h) \right\} dV' \quad (51.12)$$

We shall show that the integrand equals the k -th component of the triple vector product $[\mathbf{r} [\mathbf{j}\mathbf{r}']]$. Introducing the auxiliary notation $\mathbf{b} = [\mathbf{j}\mathbf{r}']$ and using formula (VI.33), we can write

$$\begin{aligned} [\mathbf{r}, [\mathbf{j}\mathbf{r}']]_k &= [\mathbf{r}\mathbf{b}]_k = \sum_{i,l} \varepsilon_{kil} x_i b_l = \sum_{i,l} \varepsilon_{kil} x_i \sum_{m,n} \varepsilon_{lmn} j_m x'_n \\ &= \sum_{i,l,m,n} \varepsilon_{kil} \varepsilon_{lmn} x_i j_m x'_n \end{aligned}$$

(we remind our reader that in a cyclic rearrangement of the subscripts the value of ε_{lmn} does not change). We summate over the subscript l using relation (VI.16):

$$[\mathbf{r}, [\mathbf{j}\mathbf{r}']]_k = \sum_{i,m,n} \delta_{km} \delta_{in} x_i j_m x'_n - \sum_{i,m,n} \delta_{kn} \delta_{im} x_i j_m x'_n$$

Now let us summate over the subscripts m and n . In the first sum, the addends with $m = k$ and $n = i$ are non-zero, and in the second—the addends with $n = k$ and $m = i$. Hence,

$$[\mathbf{r}, [\mathbf{j}\mathbf{r}']]_k = \sum_i x_i j_k x'_i - \sum_i x_i j_i x'_k = \sum_i x_i (j_k x'_i - j_i x'_k)$$

A comparison with (51.12) allows us to write

$$A_k = \frac{1}{2cr^3} \int_V [\mathbf{r}, [\mathbf{j}\mathbf{r}']]_k dV'$$

or in the vector form

$$\mathbf{A} = \frac{1}{2cr^3} \int_V [[\mathbf{r}']\mathbf{j}], \mathbf{r} dV' = \frac{1}{r^3} \left[\left\{ \frac{1}{2c} \int_V [\mathbf{r}']\mathbf{j} dV' \right\}, \mathbf{r} \right] \quad (51.13)$$

(we have exchanged the places of the factors in both vector products).

The quantity

$$\mathbf{m} = \frac{1}{2c} \int_V [\mathbf{r}' \mathbf{j}] dV \quad (51.14)$$

is known as the **magnetic moment** of a system¹.

For a system of discrete charges, the expression of the magnetic moment is

$$\mathbf{m} = \frac{1}{2c} \sum_a e_a [\mathbf{r}'_a \mathbf{v}_a] \quad (51.15)$$

The latter is obtained from (51.14) if we assume that

$$\mathbf{j}(\mathbf{r}') = \rho(\mathbf{r}') \mathbf{v}(\mathbf{r}') = \sum_a e_a \mathbf{v}(\mathbf{r}') \delta(\mathbf{r}' - \mathbf{r}'_a)$$

[see (41.12)] and have in mind that $\int e_a \mathbf{v}(\mathbf{r}') \delta(\mathbf{r}' - \mathbf{r}'_a) dV'$ taken in the close proximity of the point \mathbf{r}'_a transforms into $e_a \mathbf{v}_a$.

The magnetic moment depends only on the properties of a system and, as can readily be seen, does not depend on the choice of the coordinate system. Indeed, let us displace the origin of the coordinate system over the distance \mathbf{b} . The position vectors in the new system will now be $\mathbf{r}'' = \mathbf{r}' - \mathbf{b}$. The magnetic moment in the new system is

$$\mathbf{m}' = \frac{1}{2c} \int_V [\mathbf{r}'' \mathbf{j}] dV = \frac{1}{2c} \int_V [(\mathbf{r}' - \mathbf{b}) \mathbf{j}] dV = \frac{1}{2c} \int_V [\mathbf{r}' \mathbf{j}] dV - \frac{1}{2c} \int_V [\mathbf{b} \mathbf{j}] dV$$

The first term equals \mathbf{m} , and the second one can be written as

$$- \frac{1}{2c} \left[\mathbf{b}, \int \mathbf{j} dV \right]$$

But according to (51.5), $\int \mathbf{j} dV$ vanishes. Consequently, we have arrived at equality of the moments \mathbf{m}' and \mathbf{m} , Q.E.D.

With a view to (51.14), expression (51.13) can be written as follows:

$$\mathbf{A}(\mathbf{r}) = \frac{[\mathbf{m} \mathbf{r}]}{r^3} = - \left[\mathbf{m}, \nabla \frac{1}{r} \right] \quad (51.16)$$

[compare with formula (43.7)].

To find \mathbf{B} , we must calculate the curl of expression (51.16). Assuming in formula (IX.29) that $\mathbf{a} = \mathbf{m}$ and $\mathbf{b} = \mathbf{r}/r^3$, we obtain

$$\mathbf{B} = [\nabla \mathbf{A}] = \left[\nabla, \left[\mathbf{m}, \frac{\mathbf{r}}{r^3} \right] \right] = \left(\frac{\mathbf{r}}{r^3} \nabla \right) \mathbf{m} - (\mathbf{m} \nabla) \frac{\mathbf{r}}{r^3} + \mathbf{m} \left(\nabla \frac{\mathbf{r}}{r^3} \right) - \frac{\mathbf{r}}{r^3} (\nabla \mathbf{m})$$

¹ We have omitted the prime on dV because the integrand contains no unprimed coordinates. In the expressions containing both \mathbf{r}' and \mathbf{r} , the prime on dV indicated that integration is performed over the primed coordinates. In the following we shall also omit the prime on \mathbf{r} in this expression.

The vector \mathbf{m} does not depend on \mathbf{r} , therefore the first and last terms on the right-hand side vanish. According to (XI.26)

$$\nabla \frac{\mathbf{r}}{r^3} = \mathbf{r} \nabla \frac{1}{r^3} + \frac{1}{r^3} \nabla \mathbf{r} = \mathbf{r} \left(-\frac{3}{r^4} \frac{\mathbf{r}}{r} \right) + \frac{1}{r^3} \mathbf{3} = 0$$

Therefore, the third term also vanishes. Hence,

$$\mathbf{B} = -(\mathbf{m} \nabla) \frac{\mathbf{r}}{r^3} \quad (51.17)$$

By formula (XI.33), we have

$$\begin{aligned} (\mathbf{m} \nabla) \frac{\mathbf{r}}{r^3} &= \mathbf{r} \left(\mathbf{m} \cdot \nabla \frac{1}{r^3} \right) + \frac{1}{r^3} (\mathbf{m} \nabla) \mathbf{r} = -\mathbf{r} \left(\mathbf{m} \cdot \frac{3}{r^4} \frac{\mathbf{r}}{r} \right) + \frac{1}{r^3} \mathbf{m} \\ &= \frac{\mathbf{m} - 3\mathbf{e}_r (\mathbf{m} \cdot \mathbf{e}_r)}{r^3} \end{aligned}$$

[see (XI.34), \mathbf{e}_r is the unit vector of the vector \mathbf{r}]. Consequently,

$$\mathbf{B} = \frac{3\mathbf{e}_r (\mathbf{m} \cdot \mathbf{e}_r) - \mathbf{m}}{r^3} \quad (51.18)$$

A comparison with formula (43.10) shows that a magnetic field is expressed in terms of the magnetic moment by a formula like the one expressing an electric field in terms of the electrical dipole moment.

In concluding, let us calculate the magnetic moment of a current flowing through a thin wire that forms a plane loop. We choose the origin of coordinates in the plane of the loop (Fig. 51.1). According to the definition (51.14)

$$\mathbf{m} = \frac{1}{2c} \int_V [\mathbf{r} \mathbf{j}] dV$$

In the case being considered, we can make the substitution $\mathbf{j} dV = i d\mathbf{l}$ [see formulas (50.3) and (51.6)] Consequently, the expression for \mathbf{m} can be written as

$$\mathbf{m} = \frac{i}{2c} \oint [\mathbf{r}, d\mathbf{l}]$$

Designating the unit vector of a normal to the plane of the loop by the symbol \mathbf{n} (in Fig. 51.1 this unit vector is directed beyond the drawing), the integrand can be written as $\mathbf{n} r \sin \alpha d\mathbf{l}$ so that

$$\mathbf{m} = \frac{i\mathbf{n}}{c} \oint \frac{r \sin \alpha d\mathbf{l}}{2}$$

A glance at Fig. 51.1 shows that the integrand equals the area of the hatched triangle. The integral therefore equals the area f of the loop. We have thus arrived at an expression for the magnetic moment:

$$\mathbf{m} = \frac{1}{c} i f \mathbf{n}$$

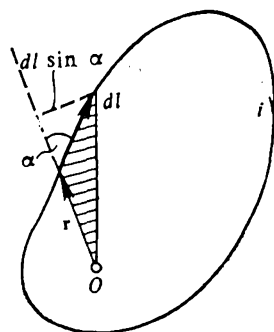


Fig. 51.1.

52. Field in Magnetics

In the presence of magnetics, the field produced by the molecules, B_{mol} , is superposed on the field of the conduction currents B_j so that the resultant microscopic magnetic field can be written as

$$B_{res} = B_j + B_{mol} \quad (52.1)$$

The field B_{res} , like the field E_{res} determined by formula (44.1), is a fast-varying function of time. In addition, it varies greatly in the space between two adjacent molecules. This is why the macroscopic quantity

$$B = B_j + \langle B_{mol} \rangle \quad (52.2)$$

is taken as a characteristic of the magnetic field in magnetics. It is called the **magnetic induction** in the magnetic. The microscopic field B_{mol} is averaged in the same way as the field E_{bound} (see Sec. 44).

It is customary practice to characterize the state of a magnetic by the magnetic moment of a unit volume of the magnetic, known as the **magnetization**. We shall denote the magnetization (which is a vector) by the symbol M . It is obvious that M can be determined as

$$M = \frac{\sum m_i}{dV} \quad (52.3)$$

where dV is an infinitely small volume, and m_i is the magnetic moment of an individual molecule; summation is performed over all the molecules contained in the volume dV .

The contribution made by a magnetic to a macroscopic magnetic field can be calculated by formula (51.16). According to (52.3), the volume element dV' of a magnetic has the magnetic moment $dm = M dV'$. Consequently, at the point determined by the position vector r , this volume element produces the magnetic potential

$$dA(r) = \frac{[M(r'), (r-r')]}{|r-r'|^3} dV' \quad (52.4)$$

[r' is the position vector of the point where the volume element dV' is, and $M(r')$ is the magnetization at this point].

The integral of expression (52.4) evaluated over the entire volume of a magnetic gives the contribution made by the magnetic to the macroscopic magnetic potential. It must be added to the magnetic potential produced by the conduction currents [see formula (48.7)]. Consequently, in the presence of a magnetic, the field is characterized by the potential

$$A(r) = \frac{1}{c} \int \frac{j(r') dV'}{|r-r'|} + \int \frac{[M(r'), (r-r')]}{|r-r'|^3} dV' = I_1 + I_2 \quad (52.5)$$

The second integral in this expression can be transformed as follows:

$$I_2 = \int \frac{[\mathbf{M}(\mathbf{r}'), (\mathbf{r} - \mathbf{r}')] }{|\mathbf{r} - \mathbf{r}'|^3} dV' = \int \left[\mathbf{M}(\mathbf{r}'), \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right] dV' \\ = \int \left[\mathbf{M}(\mathbf{r}'), \nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] dV'$$

(the prime on ∇ signifies that when taking the gradient, differentiation is performed over the primed coordinates).

Assuming in formula (XI.27) that $\varphi = 1/|\mathbf{r} - \mathbf{r}'|$ and $\mathbf{a} = \mathbf{M}(\mathbf{r}')$, we obtain

$$\left[\nabla, \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] = \left[\nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right), \mathbf{M}(\mathbf{r}') \right] + \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) [\nabla', \mathbf{M}(\mathbf{r}')]]$$

whence

$$\left[\mathbf{M}(\mathbf{r}'), \nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] = \frac{[\nabla', \mathbf{M}(\mathbf{r}')] }{|\mathbf{r} - \mathbf{r}'|} - \left[\nabla', \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right],$$

Consequently, the second term in formula (52.5) can be written as

$$I_2 = \int \frac{[\nabla', \mathbf{M}(\mathbf{r}')] }{|\mathbf{r} - \mathbf{r}'|} dV' - \int \left[\nabla', \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] dV' = I'_2 + I''_2$$

Let us transform the integral I''_2 by formula (XI.60):

$$I''_2 = \int_V \left[\nabla', \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right] dV' = \int_f \left[d\mathbf{f}', \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right]$$

If a magnetic occupies a finite volume or $\mathbf{M}(\mathbf{r}')$ diminishes sufficiently rapidly with an increasing distance from the origin of coordinates, the last integral vanishes [when a magnetic is localized in a finite volume, the integration surface can be chosen outside the magnetic, and in this case $\mathbf{M}(\mathbf{r}') = 0$ everywhere on the surface].

Hence, in the presence of magnetics, the vector potential is determined by the expression

$$\mathbf{A}(\mathbf{r}) = I_1 + I'_2 = \frac{1}{c} \int \frac{\mathbf{j}(\mathbf{r}') dV'}{|\mathbf{r} - \mathbf{r}'|} + \int \frac{[\nabla', \mathbf{M}(\mathbf{r}')] dV'}{|\mathbf{r} - \mathbf{r}'|} \\ = \frac{1}{c} \int \frac{\mathbf{j}(\mathbf{r}') + c [\nabla', \mathbf{M}(\mathbf{r}')] }{|\mathbf{r} - \mathbf{r}'|} dV' \quad (52.6)$$

The result we have obtained may be interpreted to signify that the magnetization makes the same contribution to the vector potential as the current of density

$$\mathbf{j}_M = c [\nabla \mathbf{M}] \quad (52.7)$$

(we have omitted the prime on ∇ , and accordingly \mathbf{M} is considered as a function of \mathbf{r} instead of \mathbf{r}'). It follows that in the presence of magnet-

ics, Eq. (48.3) must be written as

$$[\nabla \mathbf{B}] = \frac{4\pi}{c} \mathbf{j} + 4\pi [\nabla \mathbf{M}] \quad (52.8)$$

Combining the terms containing a curl, we obtain

$$[\nabla, (\mathbf{B} - 4\pi \mathbf{M})] = \frac{4\pi}{c} \mathbf{j}$$

The quantity

$$\mathbf{H} = \mathbf{B} - 4\pi \mathbf{M} \quad (52.9)$$

is called the **strength** of a magnetic field. It is an auxiliary macroscopic characteristic of a magnetic field similar to the electric displacement \mathbf{D} [see (45.2)]. The following equation holds for \mathbf{H} :

$$[\nabla \mathbf{H}] = \frac{4\pi}{c} \mathbf{j} \quad (52.10)$$

Experiments show that for diamagnetics and paramagnetics the magnetization is proportional to the field strength:

$$\mathbf{M} = \chi_m \mathbf{H} \quad (52.11)$$

(The magnetic is assumed to be isotropic. In addition, we have in mind fields for which the magnetization is far from saturation.)

The quantity χ_m is known as the **magnetic susceptibility** of a substance. It is positive for paramagnetics and negative for diamagnetics. Substitution of expression (52.11) into formula (52.9) yields

$$\mathbf{B} = \mathbf{H} + 4\pi \chi_m \mathbf{H} = \mu \mathbf{H} \quad (52.12)$$

where

$$\mu = 1 + 4\pi \chi_m \quad (52.13)$$

is the **permeability** of a substance.

Relations (52.11)-(52.13) are also used for ferromagnetics, treating χ_m and μ as functions of the field strength \mathbf{H} .

The expediency of introducing \mathbf{H} is due to the same considerations that were set out in Sec. 45 to substantiate the expediency of introducing \mathbf{D} .

Consider a field in a homogeneous isotropic magnetic. Assume that in the absence of the magnetic the given conduction currents produce a field characterized by the induction \mathbf{B}_0 and the potential \mathbf{A}_0 . It is known that \mathbf{B}_0 and \mathbf{A}_0 are solutions of the equations

$$[\nabla \mathbf{B}_0] = \frac{4\pi}{c} \mathbf{j} \quad (52.14)$$

$$\Delta \mathbf{A}_0 = -\frac{4\pi}{c} \mathbf{j} \quad (52.15)$$

[see (48.3) and (48.4)].

Now, without changing the conduction currents, let us fill the entire space in which the field differs from zero with a homogeneous ($\mu = \text{const}$) isotropic magnetic. The field induction will therefore become equal to \mathbf{B} , and the potential will become \mathbf{A} . Let us write equations for \mathbf{B} and \mathbf{A} . By Eq. (52.10), we have

$$[\nabla \mathbf{H}] = \left[\nabla \frac{\mathbf{B}}{\mu} \right] = \frac{4\pi}{c} \mathbf{j} \quad (52.16)$$

Substituting $[\nabla \mathbf{A}]$ for \mathbf{B} in this equation [see (47.5)] and taking into account that $\mu = \text{const}$, we can write

$$\frac{1}{\mu} [\nabla, [\nabla \mathbf{A}]] = \frac{4\pi}{c} \mathbf{j}$$

Let us develop the left-hand side of the expression obtained by formula (XI.45). The result is

$$\frac{1}{\mu} \{ \nabla (\nabla \mathbf{A}) - \Delta \mathbf{A} \} = \frac{4\pi}{c} \mathbf{j}$$

But we have agreed to choose \mathbf{A} so that $\nabla \mathbf{A}$ vanishes [see (47.7)]. Hence, the first term in braces vanishes, and we arrive at the equation

$$\Delta \frac{\mathbf{A}}{\mu} = -\frac{4\pi}{c} \mathbf{j} \quad (52.17)$$

A comparison of (52.16) with (52.14) and (52.17) with (52.15) shows that the equation for \mathbf{B}/μ coincides with that for \mathbf{B}_0 , and the equation for \mathbf{A}/μ with that for \mathbf{A}_0 . This shows that the filling of a space containing a field with a homogeneous magnetic leads to an increase in both the magnetic induction and the magnetic potential of μ times.

By Eq. (52.17), Poisson's equation (48.4) for a field in a homogeneous isotropic magnetic has the following form:

$$\Delta \mathbf{A} = -\frac{4\pi\mu}{c} \mathbf{j} \quad (52.18)$$

When a homogeneous isotropic magnetic with $\mu = \text{const}$ (i.e. with μ not depending on \mathbf{H}) fills the entire space in which the field differs from zero, the following relation holds:

$$\mathbf{H} = \frac{1}{\mu} [\nabla \mathbf{A}] = [\nabla \mathbf{A}'] \quad (52.19)$$

where $\mathbf{A}' = \mathbf{A}/\mu$.

We shall underline the fact that the field strength \mathbf{H} can be represented in the form of the curl of the function \mathbf{A}/μ only when the magnetic is homogeneous and $\mu = \text{const}$. The magnetic induction \mathbf{B} , however, can always be written as $\mathbf{B} = [\nabla \mathbf{A}]$ because $\nabla \mathbf{B} = 0$ in any conditions.

Recall that at the boundary between two magnetics, the vectors **B** and **H** must satisfy the following conditions:

$$\left. \begin{aligned} B_{n1} &= B_{n2}, & B_{\tau1}/\mu_1 &= B_{\tau2}/\mu_2 \\ H_{\tau1} &= H_{\tau2}, & \mu_1 H_{n1} &= \mu_2 H_{n2} \end{aligned} \right\} \quad (52.20)$$

A derivation of these boundary conditions can be found in textbooks of general physics.

In anisotropic magnetics, the relation between **M** and **H** is given by the expressions

$$M_i = \sum_{k=1}^3 \chi_{m, ik} H_k \quad (i = 1, 2, 3) \quad (52.21)$$

where $\chi_{m, ik}$ is a symmetric tensor of rank two called the magnetic susceptibility tensor.

The equations relating the vectors **B** and **H** can accordingly be written as follows:

$$B_i = \sum_{k=1}^3 \mu_{ik} H_k \quad (i = 1, 2, 3) \quad (52.22)$$

where the quantities

$$\mu_{ik} = \delta_{ik} + 4\pi\chi_{m, ik} \quad (52.23)$$

are the components of the magnetic susceptibility tensor (compare with the formulas in Sec. 46).

53. Law of Electromagnetic Induction

A change in the magnetic induction flux through a closed contour Γ is attended by the appearance of an induced e.m.f. in the circuit equal to

$$\mathcal{E}_i = -\frac{1}{c} \frac{d\Phi}{dt} = -\frac{1}{c} \frac{d}{dt} \int_{\Gamma} \mathbf{B} d\mathbf{f}$$

(Φ is the magnetic flux penetrating the contour). If the surface over which the integral is evaluated does not vary with time, the operations of differentiation with respect to t and integration over the coordinates may exchange their places. Hence, the expression for \mathcal{E}_i can be written as

$$\mathcal{E}_i = -\frac{1}{c} \int \frac{\partial \mathbf{B}}{\partial t} d\mathbf{f} \quad (53.1)$$

(we have written the partial derivative with respect to t because \mathbf{B} , generally speaking, is a function of both time and the coordinates).

The e.m.f. by definition is the circulation around a given contour of the field strength vector \mathbf{E}_{ext} of extraneous forces. In the present case, the strength \mathbf{E} of a vortical electric field produced by the varying magnetic field \mathbf{B} is the strength \mathbf{E}_{ext} . Consequently,

$$\mathcal{E}_i = \oint_{\Gamma} \mathbf{E}_i d\mathbf{l} = \int_{\Gamma} [\nabla \mathbf{E}] d\mathbf{f} \quad (53.2)$$

(we have used the Ostrogradsky-Gauss theorem).

Equating the right-hand sides of expressions (53.1) and (53.2), we arrive at the relation

$$\int [\nabla \mathbf{E}] d\mathbf{f} = -\frac{1}{c} \int \frac{\partial \mathbf{B}}{\partial t} d\mathbf{f} \quad (53.3)$$

Assume that both integrals are taken over the same surface [Eq. (53.3) is also observed for different surfaces if only they rest on the same contour Γ]. Relation (53.3) must be satisfied for any surface f . This is possible only if the integrand functions have the same value at every point of space. We thus arrive at the equation

$$[\nabla \mathbf{E}] = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad (53.4)$$

The equation we have obtained does not include the parameters of the contour with whose consideration we began this section. It is natural to assume that this equation must be observed for any point of a field regardless of the presence in this field of a physically separated (particularly a conducting) contour. This assumption is known to be justified experimentally.

Examination of Eq. (53.4) shows that a time-varying magnetic field \mathbf{B} must set up an electric field \mathbf{E} . Indeed, for $[\nabla\mathbf{E}]$ to be non-zero, the presence of a non-homogeneous (i.e. varying from point to point) field \mathbf{E} is needed.

54. Displacement Current

In analysing the equations describing electromagnetic phenomena, J. Maxwell gave attention to the fact that in the non-stationary case, the equation

$$[\nabla\mathbf{H}] = \frac{4\pi}{c} \mathbf{j} \quad (54.1)$$

[see (52.10)] is incompatible at $\partial\rho/\partial t \neq 0$ with the continuity equation

$$\nabla\mathbf{j} = -\frac{\partial\rho}{\partial t} \quad (54.2)$$

[see (51.1)]. To convince ourselves that this is true, let us take the divergence of both sides of Eq. (54.1). Since the divergence of a curl is always zero [see (XI.44)], we arrive at the conclusion that the divergence of \mathbf{j} and, consequently, $\partial\rho/\partial t$ too, cannot be non-zero. But the conclusion that $\partial\rho/\partial t$ always equals zero does not agree with experiments: in non-stationary processes, the density of the charges quite often varies with time.

Equations (54.1) and (54.2) can be brought into agreement by adding in (54.1) to \mathbf{j} a quantity (we shall denote it \mathbf{j}_d) having the dimension of current density. This quantity must be determined so that the condition

$$\nabla(\mathbf{j} + \mathbf{j}_d) = 0$$

will always be satisfied. It follows from this condition with a view to (54.2) that the addend \mathbf{j}_d must satisfy the relation

$$\nabla\mathbf{j}_d = -\nabla\mathbf{j} = \frac{\partial\rho}{\partial t} \quad (54.3)$$

Time differentiation of Eq. (45.3) yields the equation

$$\frac{\partial}{\partial t} (\nabla\mathbf{D}) = 4\pi \frac{\partial\rho}{\partial t}$$

or, changing the sequence of differentiating \mathbf{D} with respect to time and the coordinates,

$$\nabla \left(\frac{\partial \mathbf{D}}{\partial t} \right) = 4\pi \frac{\partial \rho}{\partial t}$$

Substituting the expression obtained from this equation for $\partial \rho / \partial t$ into formula (54.3), we arrive at the equation

$$\nabla \mathbf{j}_d = \frac{1}{4\pi} \nabla \left(\frac{\partial \mathbf{D}}{\partial t} \right)$$

It follows from this equation that the expression

$$\mathbf{j}_d = \frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t} \quad (54.4)$$

must be taken as \mathbf{j}_d [in the general case the arbitrary function of time $f(t)$ must be added to the right-hand side, but we have assumed that it equals zero].

Maxwell thus arrived at the conclusion that for a time-varying field Eq. (54.1) must be written as

$$[\nabla \mathbf{H}] = \frac{4\pi}{c} \left(\mathbf{j} + \frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t} \right)$$

or

$$[\nabla \mathbf{H}] = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \quad (54.5)$$

He called the quantity \mathbf{j}_d the density of the displacement current and the sum $\mathbf{j} + \mathbf{j}_d$ the density of the total current.

A glance at Eq. (54.5) shows that a magnetic field may be produced not only by conduction currents, but also by a time-varying electric field. Hence, the introduction of the displacement current made the fields \mathbf{E} and \mathbf{B} equivalent with respect to the ability to produce each other.

55. Maxwell's Equations

The collection of Eqs. (45.3), (47.4), (53.4) and (54.5) forms a system of Maxwell's equations. It is customary practice to group these equations in pairs. The equations

$$[\nabla \mathbf{E}] = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad (55.1)$$

and

$$\nabla \mathbf{B} = 0 \quad (55.2)$$

are known as the first pair of Maxwell's equations, and the equations

$$[\nabla \mathbf{H}] = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \quad (55.3)$$

and

$$\nabla \mathbf{D} = 4\pi\rho \quad (55.4)$$

as the second pair of equations. We must note that the first pair of equations includes only the basic characteristics of a field, namely, \mathbf{E} and \mathbf{B} . The second pair contains the auxiliary quantities \mathbf{D} and \mathbf{H} .

Maxwell's equations are the foundation of all electrodynamics. They play the same role in classical electrodynamics that Newton's equations do in classical mechanics.

The system of Maxwell's equations includes eight scalar equations [each of the vector equations (55.1) and (55.3) is equivalent to three scalar ones] containing twelve unknown scalar functions (the components of the vectors \mathbf{E} , \mathbf{B} , \mathbf{D} and \mathbf{H}). Therefore, Eqs. (55.1)-(55.4) by themselves are insufficient for determining the electromagnetic fields in a substance. In this sense, the system of Maxwell's equations is incomplete. To be able to analyse fields, Maxwell's equations must be supplemented with equations relating the quantities \mathbf{D} , \mathbf{j} and \mathbf{E} , and also \mathbf{H} and \mathbf{B} , with one another. These equations (sometimes called equations of state) have the form [see (45.5) and (52.12)]

$$\mathbf{D} = \epsilon \mathbf{E} \quad (55.5)$$

$$\mathbf{B} = \mu \mathbf{H} \quad (55.6)$$

$$\mathbf{j} = \sigma \mathbf{E} \quad (55.7)$$

(here σ is the electrical conductivity of the medium). To solve problems, we must know the characteristics of the medium, ϵ , μ , and σ , which are constants in the simplest case.

We must note that Maxwell's equations can be written so as not to contain the auxiliary quantities \mathbf{D} and \mathbf{H} . For this purpose, we must replace these quantities with their values from (45.2) and (52.9) in the second pair of equations. We invite our reader to convince himself that after such a replacement Maxwell's equations become

$$[\nabla \mathbf{E}] = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \mathbf{B} = 0 \quad (55.8)$$

$$[\nabla \mathbf{B}] = \frac{4\pi}{c} \left(\mathbf{j} + c [\nabla \mathbf{M}] + \frac{\partial \mathbf{P}}{\partial t} \right) + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla \mathbf{E} = 4\pi (\rho - \nabla \mathbf{P}) \quad (55.9)$$

To solve the system of equations (55.8), (55.9), we must know the form of the functions $\mathbf{P} = \mathbf{P}(\mathbf{E})$, $\mathbf{M} = \mathbf{M}(\mathbf{B})$, and $\mathbf{j} = \mathbf{j}(\mathbf{E})$.

In the following, we shall have to do with electromagnetic fields in homogeneous and isotropic media whose permittivity ϵ and permeability μ are constant. Here ϵ and μ can be put outside the signs of the derivatives or, conversely, can be put inside these signs. Hence, Eqs. (55.1)-(55.4) can be written as

$$[\nabla \mathbf{E}] = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \mathbf{B} = 0 \quad (55.10)$$

$$[\nabla \mathbf{B}] = \frac{4\pi\mu}{c} \mathbf{j} + \frac{\epsilon\mu}{c} \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla \mathbf{E} = \frac{4\pi}{\epsilon} \rho \quad (55.11)$$

In the absence of media that can be polarized and magnetized, ϵ and μ equal unity, so that Maxwell's equations for a field in a vacuum are as follows:

$$[\nabla \mathbf{E}] = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \mathbf{B} = 0 \quad (55.12)$$

$$[\nabla \mathbf{B}] = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla \mathbf{E} = 4\pi \rho \quad (55.13)$$

Since in the Gaussian system of units \mathbf{H} coincides with \mathbf{B} for a field in a vacuum, in this case we may write \mathbf{H} instead of \mathbf{B} in Eqs. (55.12) and (55.13).

56. Potentials of Electromagnetic Field

In Sec. 47, taking advantage of the fact that $\nabla \mathbf{B} = 0$, we wrote the magnetic induction as

$$\mathbf{B} = [\nabla \mathbf{A}] \quad (56.1)$$

where \mathbf{A} is an auxiliary function known as the vector potential. Expression (56.1) also holds for a time-varying field. In this case, however, \mathbf{A} must be considered as a function of not only the coordinates, but also the time t , namely, $\mathbf{A} = \mathbf{A}(\mathbf{r}, t)$.

Let us substitute (56.1) into Eq. (55.1). The result is

$$[\nabla \mathbf{E}] = -\frac{1}{c} \frac{\partial}{\partial t} [\nabla \mathbf{A}] = -\frac{1}{c} \left[\nabla, \frac{\partial \mathbf{A}}{\partial t} \right]$$

This relation can be written as

$$\left[\nabla, \left(\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) \right] = 0$$

Since the curl of the vector $\mathbf{E} + (1/c) \partial \mathbf{A} / \partial t$ vanishes, this vector can be written as the gradient of a function φ :

$$\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\nabla \varphi \quad (56.2)$$

The function φ is called the scalar potential of an electromagnetic field. In the non-stationary case, it depends on \mathbf{r} and on t . It can be seen from (56.2) that the potentials φ and \mathbf{A} have the same dimension.

By (56.2), we have

$$\mathbf{E} = -\nabla \varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \quad (56.3)$$

The electric field strength in the general case is thus determined not only by the scalar potential φ , but also by the vector potential \mathbf{A} . The second term in (56.3) is evidently due to electromagnetic induction. For a stationary field, $\partial \mathbf{A} / \partial t = 0$, and formula (56.3) transforms into (41.5).

Relations (56.1) and (56.3) express the magnetic and electric fields in terms of the vector and scalar potentials.

Let us find equations that will allow us to calculate the potentials A and φ for a field in a homogeneous and isotropic medium with constant ε and μ . For this purpose, we shall substitute in Eqs. (55.11) expression (56.3) for E and the curl of A for B :

$$[\nabla, [\nabla A]] = \frac{4\pi\mu}{c} \mathbf{j} - \frac{\varepsilon\mu}{c} \frac{\partial}{\partial t} (\nabla\varphi) - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 A}{\partial t^2}$$

$$\nabla (\nabla\varphi) + \frac{1}{c} \nabla \left(\frac{\partial A}{\partial t} \right) = -\frac{4\pi}{\varepsilon} \rho$$

Taking into account that $[\nabla, [\nabla A]] = \nabla (\nabla A) - \Delta A$ [see (XI.45)], and $\nabla (\nabla\varphi) = \Delta\varphi$, we can write the equations we have obtained as follows:

$$\Delta A - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 A}{\partial t^2} = -\frac{4\pi\mu}{c} \mathbf{j} + \nabla \left(\nabla A + \frac{\varepsilon\mu}{c} \frac{\partial\varphi}{\partial t} \right) \quad (56.4)$$

$$\Delta\varphi = -\frac{4\pi\rho}{\varepsilon} - \frac{1}{c} \frac{\partial}{\partial t} (\nabla A) \quad (56.5)$$

(in some terms of the equations we have changed the sequence of differentiation with respect to the coordinates and time).

Equations (56.4) and (56.5) are exactly the ones we are looking for to find A and φ . These equations are quite complicated. The fact that they are mutually related—each of them contains both A and φ —is especially unpleasant. We shall see below, however, that the potentials can be selected so that these equations become greatly simplified.

The potentials A and φ are determined non-uniquely. There is consequently a certain freedom in choosing them. Particularly, an arbitrary constant vector may be added to A , and an arbitrary constant to φ without any change in the values of B and E . Naturally, the potentials should be chosen in the most convenient way for the given case. This most expedient choice of the potentials is known as their **gauging**. We must note that we have already taken advantage of the possibility of gauging in magnetostatics: in Sec. 47 we chose A so that its divergence would be zero [see formula (47.7)].

Let us determine the most general form of the gauge transformations, i.e. such transformations of the potentials A and φ at which the fields E and B remain constant. The field $B = [\nabla A]$ will not change if we add to A the gradient of an arbitrary scalar function f (the curl of the gradient is zero), i.e. pass over from A to A' equal to $A + \nabla f$:

$$A \rightarrow A' = A + \nabla f \quad (56.6)$$

For the electric field $E = -\nabla\varphi - (1/c) \partial A / \partial t$ to remain unchanged here, simultaneously with the transition (56.6) we must perform

the transition

$$\varphi \rightarrow \varphi' = \varphi - \frac{1}{c} \frac{\partial f}{\partial t} \quad (56.7)$$

where f is the same function as in (56.6). Indeed, the field \mathbf{E}' determined by the potentials \mathbf{A}' and φ' will in this case be

$$\begin{aligned} \mathbf{E}' &= -\nabla\varphi' - \frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t} = -\nabla\varphi + \frac{1}{c} \nabla \frac{\partial f}{\partial t} \\ &\quad - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c} \frac{\partial}{\partial t} \nabla f = -\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E} \end{aligned}$$

[recall that $\nabla(\partial f/\partial t) = \partial(\nabla f)/\partial t$]. Hence, the transformation of the potentials by formulas (56.6) and (56.7) does not change the value of \mathbf{B} and \mathbf{E} .

The transformations (56.6) and (56.7) are gauge transformations of the most general kind. Since the fields \mathbf{B} and \mathbf{E} remain unchanged in these transformations, all the equations describing these fields must be invariant with respect to the gauge transformations. This invariance is called a **gauge invariance**.

In practice, as we have already mentioned, the gauge that is the most expedient in each specific case is used. Particularly, we can choose the potentials so that the following condition will be observed:

$$\nabla \mathbf{A} + \frac{\epsilon\mu}{c} \frac{\partial \varphi}{\partial t} = 0 \quad (56.8)$$

It is called the **Lorentz condition**.

For a field in a vacuum, the Lorentz condition is

$$\nabla \mathbf{A} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0 \quad (56.9)$$

We shall show that condition (56.8) can be satisfied by the proper choice of the function f in formulas (56.6) and (56.7). For this purpose, we shall introduce the values of \mathbf{A}' and φ' determined by these formulas into Eq. (56.8):

$$\nabla \mathbf{A} + \Delta f + \frac{\epsilon\mu}{c} \frac{\partial \varphi}{\partial t} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 f}{\partial t^2} = 0$$

$[\nabla(\nabla f) = \Delta f]$. Hence, we obtain an equation for finding the function f :

$$\Delta f - \frac{\epsilon\mu}{c^2} \frac{\partial^2 f}{\partial t^2} = F(\mathbf{r}, t) \quad (56.10)$$

where $F(\mathbf{r}, t) = -\nabla \mathbf{A} - (\epsilon\mu/c) \partial \varphi / \partial t$ is the preset function of \mathbf{r} and t . Introducing the function f obtained from the solution of this equation into formulas (56.6) and (56.7), we shall find the values of the potentials \mathbf{A}' and φ' satisfying condition (56.8).

The gauging of potentials satisfying condition (56.8) is called the Lorentz gauge. This gauge is in the greatest favour.

The Lorentz condition greatly restricts the set of potential values suitable for describing a given field, but it nevertheless makes the choice of the potentials quite unique. Indeed, without violating condition (56.8), we can perform the transformations

$$\left. \begin{aligned} \mathbf{A} &\rightarrow \mathbf{A}' = \mathbf{A} + \nabla\psi \\ \varphi &\rightarrow \varphi' = \varphi - \frac{1}{c} \frac{\partial\psi}{\partial t} \end{aligned} \right\} \quad (56.11)$$

(both sets of potentials— \mathbf{A} , φ and \mathbf{A}' , φ' —are assumed to satisfy the Lorentz condition), where the function ψ is the solution of the equation

$$\Delta\psi - \frac{\varepsilon\mu}{c^2} \frac{\partial^2\psi}{\partial t^2} = 0 \quad (56.12)$$

Indeed, introducing into the left-hand side of (56.8) the primed potentials from (56.11) instead of \mathbf{A} and φ , we obtain the expression

$$\nabla\mathbf{A} + \Delta\psi + \frac{\varepsilon\mu}{c} \frac{\partial\varphi}{\partial t} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2\psi}{\partial t^2} = \left(\nabla\mathbf{A} + \frac{\varepsilon\mu}{c} \frac{\partial\varphi}{\partial t} \right) + \left(\Delta\psi - \frac{\varepsilon\mu}{c^2} \frac{\partial^2\psi}{\partial t^2} \right)$$

which owing to (56.8) and (56.12) equals zero. Therefore, if the potentials \mathbf{A} and φ belong to the Lorentz gauge, the potentials \mathbf{A}' and φ' determined by transformations (56.11) [with ψ obeying (56.12)] belong to the same gauge. This allows us to impose on the potentials an additional condition besides condition (56.8). For instance, we can require that the potential φ vanish. For this purpose, according to the second of Eqs. (56.11), it is sufficient to choose the function ψ so that its time derivative will be $c\varphi$.

We can also adopt the requirement that

$$\nabla\mathbf{A} = 0 \quad (56.13)$$

as an additional condition. It follows from (56.11) that $\nabla\mathbf{A}' = \nabla\mathbf{A} + \Delta\psi$. Therefore, for the requirement that $\nabla\mathbf{A}' = 0$ to be satisfied, the equality

$$\Delta\psi = -\nabla\mathbf{A}$$

must be observed. At the same time, from (56.12), we have

$$\Delta\psi = \frac{\varepsilon\mu}{c^2} \frac{\partial^2\psi}{\partial t^2}$$

Therefore, if we take ψ the solution of the equation

$$\frac{\varepsilon\mu}{c^2} \frac{\partial^2\psi}{\partial t^2} = -\nabla\mathbf{A}$$

and introduce this solution into (56.11), we obtain values of A' and φ' satisfying both the Lorentz condition and the requirement (56.13).

The Lorentz gauge satisfying the additional condition (56.13) is called the **Coulomb** (or transverse) gauge. This is the gauge we used in Chap. IX [see (47.7)]. Examination of (56.5) shows that in the Coulomb gauge the scalar potential satisfies Poisson's equation

$$\Delta\varphi = -\frac{4\pi\rho}{\varepsilon}$$

[see Eq. (45.13)], i.e. is a Coulomb potential. This explains the origin of the name "Coulomb gauge".

57. D'Alembert's Equation

When condition (56.8) is satisfied, the last term in Eq. (56.4) vanishes. In addition, the time derivative of ∇A has the value $-(\varepsilon\mu/c) \partial^2\varphi/\partial t^2$. Consequently, Eqs. (56.4) and (56.5) become

$$\Delta A - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 A}{\partial t^2} = -\frac{4\pi\mu}{c} \mathbf{j} \quad (57.1)$$

$$\Delta\varphi - \frac{\varepsilon\mu}{c^2} \frac{\partial^2\varphi}{\partial t^2} = -\frac{4\pi\rho}{\varepsilon} \quad (57.2)$$

Hence, instead of two mutually related equations, we have obtained two independent equations, the equations for A and φ having acquired a similar form.

A differential equation of the kind

$$\Delta f - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 f}{\partial t^2} = F(\mathbf{r}, t) \quad (57.3)$$

is known as d'Alembert's equation [compare with (56.10)]. It can be written in a very compact form if we introduce the **d'Alembertian operator**

$$\square = \Delta - \frac{\varepsilon\mu}{c^2} \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2}{\partial t^2} \quad (57.4)$$

Now Eq. (57.3) becomes

$$\square f = F(\mathbf{r}, t) \quad (57.5)$$

In the stationary case, the time derivatives vanish, and D'Alembert's equation transforms into Poisson's one [see Eqs. (45.13) and (52.18)].

When we use the symbol (57.4), Eqs. (57.1) and (57.2) become

$$\square A = -\frac{4\pi\mu}{c} \mathbf{j} \quad (57.6)$$

$$\square\varphi = -\frac{4\pi\rho}{\varepsilon} \quad (57.7)$$

From a mathematical viewpoint, Eqs. (57.6) and (57.7) are simpler than Maxwell's equations. It is therefore simpler to calculate the potentials A and φ than the fields \mathbf{B} and \mathbf{E} directly. When the potentials are known, there is no great difficulty in finding the fields by formulas (56.1) and (56.3). This circumstance justifies the introduction of the auxiliary quantities A and φ . In addition, as we shall see in the following chapter, the introduction of the potentials allows us to give the equations of electrodynamics a very compact and refined form.

58. Density and Flux of Electromagnetic Field Energy

Experiments show that an electromagnetic field has an energy that propagates in space with a certain density w and can flow from one place to another. This energy can also transform into other kinds of energy, for example, it can go to do work on the particles of a substance.

Let us separate in a substance in which there is a macroscopic electromagnetic field the volume V confined by the surface f . This volume contains energy of the field equal to

$$W = \int_V w dV \quad (58.1)$$

The energy can flow outward from the volume V through the surface f confining it. If we introduce the vector \mathbf{S} of the energy flux density, the flux of energy flowing outward through f can be written as

$$\Phi = \int_f S_n df = \int_V \nabla \mathbf{S} dV \quad (58.2)$$

(we have used the Ostrogradsky-Gauss theorem).

Let us find the work done in unit time by the field forces on the particles of a substance. The forces of a field do the following work on a particle having the charge e_a and travelling at the velocity \mathbf{v}_a in unit time:

$$N_a = e_a \left\{ \mathbf{E} + \frac{1}{c} [\mathbf{v}_a \mathbf{B}] \right\} \mathbf{v}_a = e_a \mathbf{v}_a \mathbf{E}$$

(the scalar triple product $\mathbf{v}_a [\mathbf{v}_a \mathbf{B}]$ vanishes). Summating this expression over all the particles confined in unit volume, we obtain the density of the power developed by the forces of the electromagnetic field in doing work on the particles of the substance. Designating the power density by N , we can write

$$N = \sum_a N_a = \mathbf{E} \sum_{a=1}^{\infty} e_a \mathbf{v}_a$$

The sum $\Sigma e_a \mathbf{v}_a$ taken over unit volume is the density \mathbf{j} of the electric current (if all the particles are identical, travel at the same velocity, and their number in unit volume is n , the sum $\Sigma e_a \mathbf{v}_a$ transforms into the expression $\mathbf{j} = env$ known from the general course of physics). Hence,

$$N = \mathbf{E} \mathbf{j} \quad (58.3)$$

The energy W contained in the volume V may diminish because of energy flowing outward through the surface f and because of work being done on the particles of the substance. Consequently, the following relation must be observed:

$$\frac{dW}{dt} = -\Phi - \int_V N dV$$

(remember that N is the power density, i.e. power developed per unit volume). Substitution into the above equation of expressions (58.1)-(58.3) for W , Φ , and N yields

$$\frac{d}{dt} \int_V w dV = - \int_V \nabla \mathbf{S} dV - \int_V \mathbf{E} \mathbf{j} dV$$

Let us exchange the places of time differentiation and integration over the coordinates in the first integral on the right-hand side, and also combine all three integrals into one. The result is

$$\int_V \left(\frac{\partial w}{\partial t} + \nabla \mathbf{S} + \mathbf{E} \mathbf{j} \right) dV = 0 \quad (58.4)$$

(we have used the symbol of the partial derivative because w in the general case is a function not only of time, but also of the coordinates).

Condition (58.4) must be observed for any arbitrarily chosen volume V . We thus conclude that the integrand function must vanish at every point. Consequently, we arrive at a differential equation that can be written as

$$\mathbf{E} \mathbf{j} = -\frac{\partial w}{\partial t} - \nabla \mathbf{S} \quad (58.5)$$

The energy density w and the energy flux density \mathbf{S} are functions of quantities characterizing a field. To find the form of these functions, let us attempt to transform the expression for $\mathbf{E} \mathbf{j}$ so that it would become the sum of two addends, one of which would be the time derivative of a scalar quantity, which we would be able to identify with w , and the second—the divergence of a vector quantity, which we would be able to identify with \mathbf{S} .

Using Maxwell's equation (55.3), let us express \mathbf{j} in terms of the field characteristics \mathbf{H} and \mathbf{D} :

$$\mathbf{j} = \frac{c}{4\pi} [\nabla \mathbf{H}] - \frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t}$$

Now let us find the scalar product of this expression and \mathbf{E} :

$$\mathbf{E} \mathbf{j} = \frac{c}{4\pi} \mathbf{E} [\nabla \mathbf{H}] - \frac{1}{4\pi} \mathbf{E} \frac{\partial \mathbf{D}}{\partial t} \quad (58.6)$$

By formula (XI.28), we have $\nabla [\mathbf{E} \mathbf{H}] = \mathbf{H} [\nabla \mathbf{E}] - \mathbf{E} [\nabla \mathbf{H}]$, whence $\mathbf{E} [\nabla \mathbf{H}] = \mathbf{H} [\nabla \mathbf{E}] - \nabla [\mathbf{E} \mathbf{H}]$. Introducing this value into (58.6), we obtain

$$\mathbf{E} \mathbf{j} = -\frac{c}{4\pi} \nabla [\mathbf{E} \mathbf{H}] + \left\{ \frac{c}{4\pi} \mathbf{H} [\nabla \mathbf{E}] - \frac{1}{4\pi} \mathbf{E} \frac{\partial \mathbf{D}}{\partial t} \right\} \quad (58.7)$$

Using Maxwell's equation (55.1), let us substitute $-(1/c) \partial \mathbf{B} / \partial t$ for $[\nabla \mathbf{E}]$. As a result, the expression in braces becomes

$$\{\dots\} = -\frac{1}{4\pi} \left\{ \mathbf{H} \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \frac{\partial \mathbf{D}}{\partial t} \right\}$$

Finally, let us use relations (55.5) and (55.6):

$$\{\dots\} = -\frac{1}{4\pi} \left\{ \mathbf{H} \frac{\partial (\mu \mathbf{H})}{\partial t} + \mathbf{E} \frac{\partial (\varepsilon \mathbf{E})}{\partial t} \right\} = -\frac{\partial}{\partial t} \left(\frac{\varepsilon E^2}{8\pi} + \frac{\mu H^2}{8\pi} \right)$$

(we have assumed that ε and μ are time-independent).

Formula (58.7) can thus be written as follows:

$$\mathbf{E} \mathbf{j} = -\frac{\partial}{\partial t} \left(\frac{\varepsilon E^2 + \mu H^2}{8\pi} \right) - \nabla \left\{ \frac{c}{4\pi} [\mathbf{E} \mathbf{H}] \right\}$$

A comparison of the obtained relation with formula (58.5) gives for w and \mathbf{S} the expressions

$$w = \frac{\varepsilon E^2 + \mu H^2}{8\pi} \quad (58.8)$$

$$\mathbf{S} = \frac{c}{4\pi} [\mathbf{E} \mathbf{H}] \quad (58.9)$$

The vector \mathbf{S} determined by formula (58.9) is called **Poynting's vector**.

We must note that expression (58.8) includes both the proper energy of the field equal to

$$w_0 = \frac{E^2 + H^2}{8\pi}$$

and the energy spent on polarization and magnetization of the medium in producing the field.

59. Momentum of Electromagnetic Field

It follows from the existence of the pressure of light that an electromagnetic field has not only energy, but also momentum. The momentum, like the energy, can "flow" from one place to another. This process can be characterized by introducing the concepts of the flux and flux density of the momentum.

The momentum flux, unlike the energy flux, is a vector, and not a scalar. Consequently, the density of the momentum flux must be a quantity such that when multiplied by the vector $d\mathbf{f}$ (here $d\mathbf{f}$ is an element of the surface) it yields a vector. It is shown in Appendix X that the scalar product of a second-rank tensor and a vector is a vector. We thus conclude that the density of the momentum flux is a tensor. Let us designate the components of this tensor by the symbol σ_{ik} . Therefore, the flux of the i -th component of the momentum through the area $d\mathbf{f}$ is determined by the expression

$$\sum_k \sigma_{ik} df_k$$

and the flux of the momentum vector by

$$\sum_i \mathbf{e}_i \sum_k \sigma_{ik} df_k$$

In a system consisting of free charged particles and an electromagnetic field, the total momentum, which is the sum of that of the particles and that of the field, must be conserved. Hence, denoting the total momentum of the particles by the symbol \mathbf{P} , and the density of the momentum (i.e. the momentum of unit volume) of an electromagnetic field by the symbol \mathbf{g} , we can write

$$\frac{d}{dt} \left(\mathbf{P} + \int_V \mathbf{g} dV \right) = \sum_i \mathbf{e}_i \oint_f \sum_k \sigma_{ik} df_k$$

The left-hand side gives the rate of growth of the total momentum contained in the volume V , and the right-hand side—the flux of the momentum of the field flowing into the volume V through the surface f confining it. We assume that no particles intersect this surface so that no momentum is carried across it by the particles.

Let us write the above relation as follows:

$$\frac{d}{dt} \mathbf{P} = - \int_V \frac{\partial \mathbf{g}}{\partial t} dV + \sum_i \mathbf{e}_i \oint_f \sum_k \sigma_{ik} df_k \quad (59.1)$$

The rate of the change in the momentum of a particle is determined by the force acting on a particle:

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c} [\mathbf{v}\mathbf{B}]$$

Summation of this expression over the particles contained in a unit volume yields

$$\frac{d\mathbf{p}_0}{dt} = \rho \mathbf{E} + \frac{1}{c} [\mathbf{j}\mathbf{B}]$$

where \mathbf{p}_0 is the density of the momentum of the particles. Finally, integrating this expression over the entire volume of the system, we find the rate of the change in the total momentum of the particles:

$$\frac{d}{dt} \mathbf{P} = \int_V \rho \mathbf{E} dV + \frac{1}{c} \int_V [\mathbf{j}\mathbf{B}] dV \quad (59.2)$$

Let us use Maxwell's equations to exclude ρ and \mathbf{j} from this expression. Assume that the medium containing the particles and the field are homogeneous and isotropic with constant ϵ and μ . It can be seen from Eqs. (55.11) that

$$\rho = \frac{\epsilon}{4\pi} \nabla \mathbf{E}, \quad \mathbf{j} = \frac{c}{4\pi\mu} [\nabla \mathbf{B}] - \frac{\epsilon}{4\pi} \frac{\partial \mathbf{E}}{\partial t}$$

Substitution of these values into (59.2) yields

$$\frac{d}{dt} \mathbf{P} = \frac{\epsilon}{4\pi} \int \mathbf{E} \nabla \mathbf{E} dV + \frac{1}{4\pi\mu} \int [[\nabla \mathbf{B}], \mathbf{B}] dV - \frac{\epsilon}{4\pi c} \int \left[\frac{\partial \mathbf{E}}{\partial t}, \mathbf{B} \right] dV \quad (59.3)$$

We shall transform Eq. (59.3) using the relation

$$\frac{\partial}{\partial t} [\mathbf{E}\mathbf{B}] = \left[\frac{\partial \mathbf{E}}{\partial t}, \mathbf{B} \right] + \left[\mathbf{E}, \frac{\partial \mathbf{B}}{\partial t} \right]$$

whence

$$\left[\frac{\partial \mathbf{E}}{\partial t}, \mathbf{B} \right] = \frac{\partial}{\partial t} [\mathbf{E}\mathbf{B}] - \left[\mathbf{E}, \frac{\partial \mathbf{B}}{\partial t} \right]$$

We substitute $-c [\nabla \mathbf{E}]$ for $\partial \mathbf{B} / \partial t$ here according to the first of the equations (55.10). The result is

$$\left[\frac{\partial \mathbf{E}}{\partial t}, \mathbf{B} \right] = \frac{\partial}{\partial t} [\mathbf{E}\mathbf{B}] + c [\mathbf{E}, [\nabla \mathbf{E}]]$$

Using the value obtained in (59.3), we arrive at the expression

$$\begin{aligned} \frac{d}{dt} \mathbf{P} = & \frac{\epsilon}{4\pi} \int \mathbf{E} \nabla \mathbf{E} dV + \frac{1}{4\pi\mu} \int [[\nabla \mathbf{B}], \mathbf{B}] dV \\ & - \frac{\epsilon}{4\pi c} \int \frac{\partial}{\partial t} [\mathbf{E}\mathbf{B}] dV - \frac{\epsilon}{4\pi} \int [\mathbf{E}, [\nabla \mathbf{E}]] dV \end{aligned}$$

Let us make the substitution $\mathbf{B} = \mu \mathbf{H}$ in the second and third integrals and, in addition, exchange the places of the multipliers in the

second integral (which will cause the sign to be reversed):

$$\frac{d}{dt} \mathbf{P} = \frac{\varepsilon}{4\pi} \int \mathbf{E} \nabla \mathbf{E} dV - \frac{\mu}{4\pi} \int [\mathbf{H}, [\nabla \mathbf{H}]] dV - \frac{\varepsilon\mu}{4\pi c} \int \frac{\partial}{\partial t} [\mathbf{E} \mathbf{H}] dV - \frac{\varepsilon}{4\pi} \int [\mathbf{E}, [\nabla \mathbf{E}]] dV$$

To make this expression symmetric with respect to \mathbf{E} and \mathbf{H} , let us add $(\mu/4\pi) \int \mathbf{H} \nabla \mathbf{H} dV$ to its right-hand side. This will not change the expression because this term is zero [$\mu \nabla \mathbf{H} = \nabla (\mu \mathbf{H}) = \nabla \mathbf{B}$ is zero everywhere]. Finally, grouping the terms properly, we obtain

$$\frac{d}{dt} \mathbf{P} = -\frac{\varepsilon\mu}{4\pi c} \int \frac{\partial}{\partial t} [\mathbf{E} \mathbf{H}] dV + \frac{1}{4\pi} \int \{ \varepsilon \mathbf{E} \nabla \mathbf{E} + \mu \mathbf{H} \nabla \mathbf{H} - \varepsilon [\mathbf{E}, [\nabla \mathbf{E}]] - \mu [\mathbf{H}, [\nabla \mathbf{H}]] \} dV \quad (59.4)$$

The second integral, as we shall see below, can be transformed into an integral over the surface f confining the volume V . Hence,

$$\frac{d}{dt} \mathbf{P} = - \int_V \frac{\partial}{\partial t} \left\{ \frac{\varepsilon\mu}{4\pi c} [\mathbf{E} \mathbf{H}] \right\} dV + \text{integral over surface } f \quad (59.5)$$

A comparison of the found relation with (59.1) allows us to draw the conclusion that the density of the momentum of an electromagnetic field is determined by the expression

$$\mathbf{g} = \frac{\varepsilon\mu}{4\pi c} [\mathbf{E} \mathbf{H}] \quad (59.6)$$

Taking (58.9) into account, we can write that

$$\mathbf{g} = \frac{\varepsilon\mu}{c^2} \mathbf{S} \quad (59.7)$$

where \mathbf{S} is the Poynting vector. For a vacuum, this relation is

$$\mathbf{g} = \frac{1}{c^2} \mathbf{S} \quad (59.8)$$

We must note that expression (59.7), in addition to the proper momentum of a field, includes the momentum of the bound charges entering the composition of the medium in which the field has been produced. To obtain the momentum of the field alone, we must understand \mathbf{P} in formula (59.1) to be the mechanical momentum not only of the free charges, but also of the bound ones. Now we would have to take not the averaged macroscopic field, but the microscopic field as \mathbf{E} and \mathbf{B} and correspondingly use Maxwell's equations for a field in a vacuum in the transformations¹. This is equivalent to

¹ We did not do this from the very beginning to obtain a more general expression for the Maxwell stress tensor (see below) that is also suitable for the field in a medium.

assuming in all the formulas of the present section that $\epsilon = \mu = 1$. As a result, we would arrive at formula (59.8). Consequently, the density of the momentum of only a field in all cases (both in a vacuum and in a substance) is determined by formula (59.8). A comparison of this formula with formula (40.36) shows that there is exactly the same relation between the densities of the energy and momentum fluxes for an electromagnetic field as we obtained in Sec. 40 for an arbitrary system.

Now let us consider the second integral in formula (59.4), i.e. the integral

$$\frac{1}{4\pi} \int \{ \epsilon \mathbf{E} \nabla \mathbf{E} + \mu \mathbf{H} \nabla \mathbf{H} - \epsilon [\mathbf{E}, [\nabla \mathbf{E}]] - \mu [\mathbf{H}, [\nabla \mathbf{H}]] \} dV \quad (59.9)$$

We shall attempt to transform it into a surface integral.

The integrand contains two similar expressions of the kind

$$\mathbf{a} \nabla \mathbf{a} - [\mathbf{a}, [\nabla \mathbf{a}]]$$

One of them contains \mathbf{E} instead of \mathbf{a} , and the other \mathbf{H} .

Assuming in the formula

$$\nabla (\mathbf{a} \mathbf{b}) = [\mathbf{a}, [\nabla \mathbf{b}]] + [\mathbf{b}, [\nabla \mathbf{a}]] + (\mathbf{a} \nabla) \mathbf{b} + (\mathbf{b} \nabla) \mathbf{a}$$

[see (XI.37)] that $\mathbf{b} = \mathbf{a}$, we obtain

$$\nabla a^2 = 2 [\mathbf{a}, [\nabla \mathbf{a}]] + 2 (\mathbf{a} \nabla) \mathbf{a} \quad (59.10)$$

Let us find the value of the expression $(\nabla \mathbf{a}) \mathbf{b}$ in which it is assumed that ∇ acts on both factors following it. According to the general rule for calculating such expressions, we have

$$(\nabla \mathbf{a}) \mathbf{b} = (\nabla_a \mathbf{a}) \mathbf{b} + (\nabla_b \mathbf{a}) \mathbf{b} = \mathbf{b} \nabla \mathbf{a} + (\mathbf{a} \nabla) \mathbf{b}$$

or, assuming that $\mathbf{b} = \mathbf{a}$,

$$(\nabla \mathbf{a}) \mathbf{a} = \mathbf{a} \nabla \mathbf{a} + (\mathbf{a} \nabla) \mathbf{a}$$

After finding $(\mathbf{a} \nabla) \mathbf{a}$ from the last expression and introducing its value into formula (59.10), we obtain

$$\nabla a^2 = 2 [\mathbf{a}, [\nabla \mathbf{a}]] + 2 (\nabla \mathbf{a}) \mathbf{a} - 2 \mathbf{a} \nabla \mathbf{a}$$

whence

$$\mathbf{a} \nabla \mathbf{a} - [\mathbf{a}, [\nabla \mathbf{a}]] = (\nabla \mathbf{a}) \mathbf{a} - \frac{1}{2} \nabla a^2$$

Applying such a transformation to the integral (59.9), we can write it as follows:

$$\frac{1}{4\pi} \int \left\{ \epsilon (\nabla \mathbf{E}) \cdot \mathbf{E} + \mu (\nabla \mathbf{H}) \cdot \mathbf{H} + \frac{1}{2} \nabla (\epsilon E^2 + \mu H^2) \right\} dV$$

In this integral, the operator ∇ acts on all the functions following it. Consequently, with the aid of the transformation

$$dV \cdot \nabla \rightarrow df$$

[see (XI.65)], we can transform this integral into a surface one:

$$\frac{1}{4\pi} \oint \left\{ \epsilon \mathbf{E} (\mathbf{E} d\mathbf{f}) + \mu \mathbf{H} (\mathbf{H} d\mathbf{f}) - \frac{1}{2} (\epsilon E^2 + \mu H^2) d\mathbf{f} \right\} \quad (59.11)$$

[since $d\mathbf{f}$ does not have the properties of a differential operator, we can exchange the places of the factors in expressions such as $(d\mathbf{f} \mathbf{a}) \mathbf{a}$].

Let us develop expression (59.11) using the components of the vectors in it. The result is

$$\frac{1}{4\pi} \sum_i \mathbf{e}_i \oint \left\{ \epsilon E_i \sum_k E_k df_k + \mu H_i \sum_k H_k df_k - \frac{1}{2} (\epsilon E^2 + \mu H^2) df_i \right\}$$

Let us write df_i in the last term as $\sum_k \delta_{ik} df_k$ and factor out df_k . Let us also substitute D_i for expressions of the kind ϵE_i , and B_i for μH_i . As a result, we arrive at the expression

$$\frac{1}{4\pi} \sum_i \mathbf{e}_i \oint \sum_k \left\{ E_i D_k + H_i B_k - \frac{1}{2} (\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B}) \delta_{ik} \right\} df_k$$

that coincides with the last term of formula (59.1) if we assume that

$$\sigma_{ik} = \frac{1}{4\pi} \left\{ E_i D_k + H_i B_k - \frac{1}{2} (\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B}) \delta_{ik} \right\} \quad (59.12)$$

We established at the beginning of this section that the tensor σ_{ik} characterizes the density of the momentum flux (see also Sec. 40). The tensor σ_{ik} whose components are determined by formula (59.12) is called the **Maxwell stress tensor**.

To underline the symmetrical nature of the tensor σ_{ik} , its components are sometimes written as

$$\sigma_{ik} = \frac{1}{8\pi} \{ E_i D_k + E_k D_i + H_i B_k + H_k B_i - (\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B}) \delta_{ik} \}$$

For a field in a vacuum, formula (59.12) is simplified as follows:

$$\sigma_{ik} = \frac{1}{4\pi} \left\{ E_i E_k + B_i B_k - \frac{1}{2} (E^2 + B^2) \delta_{ik} \right\} \quad (59.13)$$

The tensor σ_{ik} allows us to reduce the problem of finding the force acting on a certain volume of a substance in an electromagnetic field to the calculation of the surface integral

$$\sum_i \mathbf{e}_i \oint \sum_k \sigma_{ik} df_k$$

Chapter XI

EQUATIONS OF ELECTRODYNAMICS IN THE FOUR-DIMENSIONAL FORM

60. Four-Potential ¹

According to the principle of relativity, the equations of electrodynamics, like all other equations expressing nature's laws, must be relativistically invariant, i.e. retain their form in Lorentz transformations (when passing from one inertial reference frame to another). Direct verification shows that Maxwell's equations meet this requirement. We shall choose a different way, however—we shall show that the equations of electrodynamics can be written in the four-dimensional form as relations between four-vectors and four-tensors, whence their relativistic invariance will follow.

Our starting point will be the thesis adopted in electrodynamics (in accordance with experiments) that the electric charge is invariant, i.e. that the magnitude of the charge of a particle is the same in all inertial reference frames. It thus follows that the quantity ρdV is also invariant:

$$\rho dV = \rho dx^1 dx^2 dx^3 = \text{inv} \quad (60.1)$$

We know that the three-dimensional volume dV is not invariant [see formula (35.17)]. We thus conclude from (60.1) that the charge density ρ is also not invariant, but changes in a transition from one reference frame to another according to a definite law. To establish this law, we shall take into consideration that a four-dimensional volume is invariant:

$$dV^* = dx^0 dx^1 dx^2 dx^3 = c dt dV = \text{inv} \quad (60.2)$$

Indeed, in passing over to another reference frame, dt and dV transform by the formulas

$$dt' = \frac{dt}{\sqrt{1-v^2/c^2}}, \quad dV' = dV \sqrt{1-v^2/c^2}$$

so that $dt' dV' = dt dV$.

¹ In this chapter, we treat fields in a vacuum, i.e. we assume that $\epsilon = 1$ and $\mu = 1$.

We advise our reader to look through Appendix XII and Chap. VII before beginning to read this chapter.

A comparison of (60.1) and (60.2) shows that ρ transforms according to the same law as dx^0 , i.e. like the time component of a four-vector.

It is shown in Appendix XII that the contravariant components of the Hamiltonian four-operator are

$$\nabla^{*\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right)$$

[see formula (XII.45)].

By (XII.38), the symbolic square of the vector ∇^* is

$$\nabla^{*2} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \quad (60.3)$$

where Δ is the three-dimensional Laplacian operator.

A comparison of (60.3) and (57.4) shows that the d'Alembertian operator for a field in a vacuum ($\varepsilon = 1$, $\mu = 1$) differs from ∇^{*2} only in its sign:

$$\nabla^{*2} = -\square \quad (60.4)$$

It follows from everything said above that Eq. (57.7) can be written as

$$\nabla^{*2}\varphi = 4\pi\rho \quad (60.5)$$

We have ascertained above that ρ has the properties of the time component of a four-vector, and ∇^{*2} , like the square of any four-vector, behaves in Lorentz transformations like an invariant. Taking this into account, we conclude on the basis of (60.5) that the potential φ must transform according to the same law as ρ , i.e. as a four-vector time component.

Consider the current density vector $\mathbf{j} = \rho\mathbf{v}$. Its components are

$$j_k = \rho v_k = \rho \frac{dx^k}{dt} \quad (k=1, 2, 3) \quad (60.6)$$

In transformations, ρ behaves like ct or $c dt$. Consequently, j_k will behave like dx^k (where $k = 1, 2, 3$), i.e. like space components of a four-vector.

Therefore, in transformations of the coordinates, ρ behaves like a four-vector time component, and the quantities j_k like four-vector space components. Hence, multiplying ρ by the scalar c (to obtain a quantity of the same dimension as j_k has), we can combine ρ and \mathbf{j} into a single four-vector called the charge-current or simply the current four-vector. Its components are

$$j^0 = c\rho, \quad j^1 = j_x, \quad j^2 = j_y, \quad j^3 = j_z$$

which can be written briefly as follows:

$$\mathbf{j}^\mu = (c\rho, \mathbf{j}) \quad (60.7)$$

We must note that the component $j^0 = c\rho$ can be written like the components (60.6) as

$$j^0 = \rho \frac{dx^0}{dt}$$

($x^0 = ct$). Therefore, the components of the current four-vector can be determined as follows:

$$j^\mu = \rho \frac{dx^\mu}{dt} \quad (\mu = 0, 1, 2, 3) \quad (60.8)$$

The continuity equation following from charge conservation is [see (51.1)]

$$\nabla \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$$

With a view to (60.7), this equation can be written in the four-dimensional form:

$$\sum_{\mu=0}^3 \frac{\partial j^\mu}{\partial x^\mu} = 0 \quad (60.9)$$

The left-hand side of relation (60.9) is the four-dimensional divergence of the current four-vector. Indeed, by analogy with the three-dimensional divergence, the four-divergence of the vector a^μ must be determined as the scalar product of the vectors ∇^* and a^μ , i.e. as

$$\sum_{\mu=0}^3 \nabla^{*\mu} a_\mu = \sum_{\mu=0}^3 \nabla_\mu^* a^\mu = \sum_{\mu=0}^3 \frac{\partial a^\mu}{\partial x_\mu^*}$$

[see expression (XII.42) for the covariant components of the vector ∇^*].

That the four-divergence of the vector j^μ equals zero is an analytical expression of the law of charge conservation. The fact that the vector (60.7) which we have introduced satisfies such a simple condition is another argument in favour of combining ρ and \mathbf{j} into a single four-vector.

Having in view that $\square = -\nabla^{*2}$, let us write Eqs. (57.6) and (57.7) as follows:

$$\nabla^{*2} \mathbf{A} = \frac{4\pi}{c} \mathbf{j} \quad (60.10)$$

$$\nabla^{*2} \varphi = \frac{4\pi}{c} (c\rho) \quad (60.11)$$

Examination of Eq. (60.10) shows that the quantities A_k behave in the same way as the quantities j_k , i.e. like space components of a

four-vector. This circumstance allows us to combine φ and \mathbf{A} into a single four-vector:

$$A^\mu = (\varphi, \mathbf{A}) \quad (60.12)$$

known as the four-potential of an electromagnetic field. Equations (60.10) and (60.11) can therefore be written as a single equation:

$$\nabla^{*2} A^\mu = \frac{4\pi}{c} j^\mu \quad (60.13)$$

or

$$\square A^\mu = -\frac{4\pi}{c} j^\mu \quad (\mu = 0, 1, 2, 3) \quad (60.14)$$

The covariant components of the four-potential are as follows:

$$A_\mu = (\varphi, -\mathbf{A}) \quad (60.15)$$

We remind our reader that the four-potential is determined non-uniquely. By (47.6), the values of the space components A_k can be replaced with the quantities

$$\bar{A}_k = A_k + \frac{\partial \psi(x^1, x^2, x^3)}{\partial x^k} \quad (60.16)$$

and the value of the component A_0 with the quantity

$$\bar{A}_0 = A_0 + C \quad (60.17)$$

[C is a constant; see the text following formula (41.5)] without changing the field characteristics \mathbf{B} and \mathbf{E} .

The Lorentz gauge condition [see (56.8)] in the four-dimensional form is

$$\sum_{\mu=0}^3 \frac{\partial A^\mu}{\partial x^\mu} = 0 \quad (60.18)$$

This signifies that the four-potential is chosen so that its four-divergence vanishes [compare with (47.7)].

61. Electromagnetic Field Tensor

Let us go over from potentials to the force characteristics \mathbf{E} and \mathbf{B} of a field. This transition is performed by formulas (56.1) and (56.3). For convenience, we shall repeat these formulas:

$$\mathbf{B} = [\nabla \mathbf{A}] \quad (61.1)$$

$$\mathbf{E} = -\nabla \varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \quad (61.2)$$

Let us write the expressions for the components of the vector **E**:

$$\begin{aligned} E_x &= -\frac{\partial\varphi}{\partial x} - \frac{1}{c} \frac{\partial A_x}{\partial t} \\ E_y &= -\frac{\partial\varphi}{\partial y} - \frac{1}{c} \frac{\partial A_y}{\partial t} \\ E_z &= -\frac{\partial\varphi}{\partial z} - \frac{1}{c} \frac{\partial A_z}{\partial t} \end{aligned}$$

In four-dimensional symbols, these formulas can be written as

$$\left. \begin{aligned} E_x &= \frac{\partial A_1}{\partial x^0} - \frac{\partial A_0}{\partial x^1} = \nabla_0^* A_1 - \nabla_1^* A_0 \\ E_y &= \frac{\partial A_2}{\partial x^0} - \frac{\partial A_0}{\partial x^2} = \nabla_0^* A_2 - \nabla_2^* A_0 \\ E_z &= \frac{\partial A_3}{\partial x^0} - \frac{\partial A_0}{\partial x^3} = \nabla_0^* A_3 - \nabla_3^* A_0 \end{aligned} \right\} \quad (61.3)$$

[see formulas (XII.42) for the covariant components of the gradient four-operator and formulas (60.15) for the covariant components of the four-potential. By the last formulas, for instance, $A_x = -A_1$, etc.].

Now let us write the expressions for the components of the vector **B**. According to (61.1)

$$\left. \begin{aligned} B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_2}{\partial x^3} - \frac{\partial A_3}{\partial x^2} = \nabla_3^* A_2 - \nabla_2^* A_3 \\ B_y &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{\partial A_3}{\partial x^1} - \frac{\partial A_1}{\partial x^3} = \nabla_1^* A_3 - \nabla_3^* A_1 \\ B_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1} = \nabla_2^* A_1 - \nabla_1^* A_2 \end{aligned} \right\} \quad (61.4)$$

It is known from tensor algebra that expressions of the kind $a_\mu b_\nu$ — $a_\nu b_\mu$ are covariant components of an antisymmetric second-rank tensor (a_μ and b_ν are covariant components of arbitrary vectors). It follows from formulas (61.3) and (61.4) that the components of the vectors **E** and **B** can be interpreted as covariant components of the antisymmetric four-tensor

$$F_{\mu\nu} = \nabla_\mu^* A_\nu - \nabla_\nu^* A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \quad (61.5)$$

The latter is called the **electromagnetic field tensor**.

Raising the indices μ and ν on both sides of (61.5) and lowering them on the right side, we get an expression for the contravariant components of the electromagnetic field tensor:

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \quad (61.6)$$

A comparison of formulas (61.3) and (61.4) with expression (61.5) shows that

$$\left. \begin{aligned} E_x = F_{01} = -F_{10}, \quad E_y = F_{02} = -F_{20}, \quad E_z = F_{03} = -F_{30} \\ B_x = F_{32} = -F_{23}, \quad B_y = F_{13} = -F_{31}, \quad B_z = F_{21} = -F_{12} \end{aligned} \right\} \quad (61.7)$$

The tensor $F_{\mu\nu}$ can thus be written as follows:

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \quad (61.8)$$

Taking into account formulas (XII.58), let us write the values of the contravariant components of the electromagnetic field tensor:

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (61.9)$$

Hence, in four-dimensional space, an electromagnetic field is described with the aid of one antisymmetric second-rank tensor instead of two vectors (\mathbf{E} and \mathbf{B}).

We must note that whereas the four-potential A^μ is determined non-uniquely, the components of the tensor $F_{\mu\nu}$ are unique. Indeed, let us replace the components A_μ in (61.5) with the quantities \bar{A}_μ according to formulas (60.16) and (60.17). If μ and ν equal 1, 2, and 3, we obtain

$$\begin{aligned} \bar{F}_{\mu\nu} &= \frac{\partial \bar{A}_\nu}{\partial x^\mu} - \frac{\partial \bar{A}_\mu}{\partial x^\nu} = \frac{\partial A_\nu}{\partial x^\mu} + \frac{\partial^2 \psi(x^1, x^2, x^3)}{\partial x^\mu \partial x^\nu} \\ &\quad - \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial^2 \psi(x^1, x^2, x^3)}{\partial x^\nu \partial x^\mu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} = F_{\mu\nu} \end{aligned}$$

If one of the indices (for instance μ) is zero, we get the relation

$$\begin{aligned} \bar{F}_{0\nu} &= \frac{\partial \bar{A}_\nu}{\partial x^0} - \frac{\partial \bar{A}_0}{\partial x^\nu} = \frac{\partial A_\nu}{\partial x^0} + \frac{\partial}{\partial x^0} \left(\frac{\partial \psi(x^1, x^2, x^3)}{\partial x^\nu} \right) \\ &\quad - \frac{\partial A_0}{\partial x^\nu} - \frac{\partial C}{\partial x^\nu} = \frac{\partial A_\nu}{\partial x^0} - \frac{\partial A_0}{\partial x^\nu} F_{0\nu} \end{aligned}$$

We have thus proved that the quantities $F_{\mu\nu}$ are determined uniquely.

62. Field Transformation Formulas

Formulas for transforming the components of an antisymmetric four-tensor are established in Appendix XII [see formulas (XII.62)]. Substitution of the values (35.8) and (35.9) for α_0 and α_1 results in these formulas becoming

$$\left. \begin{aligned} A'^{01} &= A^{01}, & A'^{02} &= \frac{A^{02} - \beta A^{12}}{\sqrt{1 - \beta^2}}, & A'^{03} &= \frac{A^{03} - \beta A^{13}}{\sqrt{1 - \beta^2}} \\ A'^{12} &= \frac{A^{12} - \beta A^{02}}{\sqrt{1 - \beta^2}}, & A'^{13} &= \frac{A^{13} - \beta A^{03}}{\sqrt{1 - \beta^2}}, & A'^{23} &= A^{23} \end{aligned} \right\} \quad (62.1)$$

(here $\beta = v_0/c$).

Writing formulas (62.1) for the tensor (61.9), we obtain

$$\begin{aligned} F'^{01} &= F^{01}, & \text{i.e. } E'_x &= E_x, \\ F'^{02} &= \frac{F^{02} - \beta F^{12}}{\sqrt{1 - \beta^2}}, & \text{i.e. } E'_y &= \frac{E_y - \beta B_z}{\sqrt{1 - \beta^2}} \end{aligned}$$

etc. Writing out relations (62.1) for all the components of the tensor $F^{\mu\nu}$ and replacing $F^{\mu\nu}$ with the relevant values of E_k and B_k , we arrive at formulas for transforming the components of the vectors **E** and **B** in a transition from one inertial reference frame to another:

$$\left. \begin{aligned} E'_x &= E_x, & E'_y &= \frac{E_y - \beta B_z}{\sqrt{1 - \beta^2}}, & E'_z &= \frac{E_z + \beta B_y}{\sqrt{1 - \beta^2}} \\ B'_x &= B_x, & B'_y &= \frac{B_y + \beta E_z}{\sqrt{1 - \beta^2}}, & B'_z &= \frac{B_z - \beta E_y}{\sqrt{1 - \beta^2}} \end{aligned} \right\} \quad (62.2)$$

The formulas for the inverse transformation differ from these formulas only in the sign of the terms containing the factor β (i.e. in the sign of v_0).

Resolving the vectors **E** and **B** into components parallel to the x -axis (and, consequently, to the vector v_0) and components perpendicular to this axis (i.e. writing, for example, **E** as $E_{\parallel} + E_{\perp}$), we can write formulas (62.2) in the vector form¹:

$$\left. \begin{aligned} E'_{\parallel} &= E_{\parallel}, & E'_{\perp} &= \frac{E_{\perp} + (1/c) [v_0 B_{\perp}]}{\sqrt{1 - v_0^2/c^2}} \\ B'_{\parallel} &= B_{\parallel}, & B'_{\perp} &= \frac{B_{\perp} - (1/c) [v_0 E_{\perp}]}{\sqrt{1 - v_0^2/c^2}} \end{aligned} \right\} \quad (62.3)$$

¹ We must note that since B_{\parallel} and v_0 are collinear, $[v_0 B] = [v_0 B_{\parallel}] + [v_0 B_{\perp}] = [v_0 B_{\perp}]$. Similarly, $[v_0 E] = [v_0 E_{\perp}]$. For this reason in the vector products of formulas (62.3), the subscript " \perp " at B_{\perp} and E_{\perp} may be discarded.

(recall that $v_x = v_0$, and $v_y = v_z = 0$). A glance at these formulas shows that the longitudinal¹ components of the fields in a transition from one reference frame to another do not change, and only the lateral components are transformed.

If $\beta \ll 1$ (i.e. $v_0 \ll c$), the expression $1/\sqrt{1-\beta^2}$ approximately equals $1 + \frac{1}{2}\beta^2$. Consequently, within terms of the order of $\beta = v_0/c$, formulas (62.2) can be written as follows:

$$\begin{aligned} E'_x &= E_x, & E'_y &= E_y - (v_0/c) B_z, & E'_z &= E_z + (v_0/c) B_y \\ B'_x &= B_x, & B'_y &= B_y + (v_0/c) E_z, & B'_z &= B_z - (v_0/c) E_y \end{aligned} \quad (62.4)$$

It is not difficult to see that these formulas can be written in the vector form:

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c} [\mathbf{v}_0 \mathbf{B}], \quad \mathbf{B}' = \mathbf{B} - \frac{1}{c} [\mathbf{v}_0 \mathbf{E}] \quad (62.5)$$

[compare with formulas (62.3)].

If there is only an electric field in the frame K (i.e. $\mathbf{E} \neq 0$ and $\mathbf{B} = 0$), both fields exist in the frame K' . By formulas (62.5), these fields are

$$\mathbf{E}' = \mathbf{E}, \quad \mathbf{B}' = -\frac{1}{c} [\mathbf{v}_0 \mathbf{E}]$$

Having in view that $\mathbf{E} = \mathbf{E}'$, we can write that

$$\mathbf{B}' = -\frac{1}{c} [\mathbf{v}_0 \mathbf{E}'] \quad (62.6)$$

Relation (62.6)² indicates that the fields \mathbf{B}' and \mathbf{E}' are mutually perpendicular. It can also be seen from this relation that the field \mathbf{B}' is perpendicular to the vector \mathbf{v}_0 , i.e. to the x -axis.

It can be shown similarly that when there is only a magnetic field in the frame K , the vectors \mathbf{B}' and \mathbf{E}' are related by the expression

$$\mathbf{E}' = \frac{1}{c} [\mathbf{v}_0 \mathbf{B}'] \quad (62.7)$$

Consequently, in this case too the fields \mathbf{B}' and \mathbf{E}' are mutually perpendicular. In addition, \mathbf{E}' is perpendicular to \mathbf{v}_0 , i.e. to the x -axis.

Hence, if only one of the fields (\mathbf{E} or \mathbf{B}) exists in the frame K , in any other frame K' the fields \mathbf{B}' and \mathbf{E}' are mutually perpendicular. The opposite conclusion also holds: if the fields \mathbf{B} and \mathbf{E} are mutually perpendicular in a frame K (and the magnitudes of

¹ For simplicity, we shall use this name for the components parallel to the vector of the relative velocity of the frames K and K' (to the vector \mathbf{v}_0). We shall call the perpendicular components lateral ones.

² We invite our reader to convince himself that the same relation between the vector \mathbf{B}' and \mathbf{E}' can also be obtained in the case being considered from the accurate formulas (62.2).

the fields satisfy the conditions indicated below), frames K' exist in which the field is purely electric, and also frames in which the field is purely magnetic. Let us find the velocities of the relevant systems beginning with a treatment of the case when $v_0 \ll c$.

Assume that the vectors \mathbf{B} and \mathbf{E} are mutually perpendicular. It follows from formulas (62.5) that for the field in the frame K' to be purely electric (i.e. for \mathbf{B}' to vanish), the vector \mathbf{v}_0 must satisfy the condition

$$\mathbf{B} = \frac{1}{c} [\mathbf{v}_0 \mathbf{E}]$$

This condition will be observed if the vector \mathbf{v}_0 is perpendicular to \mathbf{B} (the vector \mathbf{E} is perpendicular to \mathbf{B} according to our assumption) and, in addition, $v_0 E \sin \alpha = cB$, where α is the angle between the vectors \mathbf{v}_0 and \mathbf{E} . Hence, the field is purely electric in all frames travelling in directions perpendicular to \mathbf{B} provided that the velocity v_0 of the given frame is $cB/E \sin \alpha$. Since the velocity v_0 of the system cannot exceed c , the reference frames being considered exist only provided that $cB < E \sin \alpha$. If $E \leq cB$, this condition is not observed at any angle α . Therefore, in this case, notwithstanding the mutual perpendicularity of \mathbf{B} and \mathbf{E} , no frames exist in which the field is purely electric.

It is not difficult to see that the result we have obtained is true without stipulating that $v_0 \ll c$. For this purpose, let us turn to formulas (62.3). Assume that \mathbf{B} and \mathbf{E} are mutually perpendicular. Let us take the frame K' whose velocity \mathbf{v}_0 is perpendicular to \mathbf{B} and equal in magnitude to $cB/E \sin \alpha$ (here α is the angle between the vectors \mathbf{v}_0 and \mathbf{E}). Since the vectors \mathbf{v}_0 and \mathbf{B} are mutually perpendicular, the component B_{\parallel} vanishes. By formulas (62.3), B'_{\parallel} also vanishes. Let us consider the numerator of formula (62.3) for \mathbf{B}'_{\perp} . With the direction of \mathbf{v}_0 we have chosen, the vector \mathbf{B}_{\perp} equals \mathbf{B} . The vector product of the vectors \mathbf{v}_0 and \mathbf{E} can be written as

$$[\mathbf{v}_0 \mathbf{E}] = [\mathbf{v}_0 \mathbf{E}_{\parallel}] + [\mathbf{v}_0 \mathbf{E}_{\perp}] = [\mathbf{v}_0 \mathbf{E}_{\perp}]$$

(the first term vanishes because the vectors \mathbf{v}_0 and \mathbf{E}_{\parallel} are collinear). Consequently, the formula for \mathbf{B}'_{\perp} for the case being considered can be written as follows:

$$\mathbf{B}'_{\perp} = \frac{\mathbf{B} - (1/c) [\mathbf{v}_0 \mathbf{E}]}{\sqrt{1 - v_0^2/c^2}} \quad (62.8)$$

According to our condition, \mathbf{B} is perpendicular to both \mathbf{E} and \mathbf{v}_0 . Therefore, the vector $[\mathbf{v}_0 \mathbf{E}]$ is collinear to the vector \mathbf{B} . By properly choosing the direction of the vector \mathbf{v}_0 (to the right or the left), we can make the vectors $[\mathbf{v}_0 \mathbf{E}]$ and \mathbf{B} have the same direction. The numerator of formula (62.8) thus contains the difference of two identically directed vectors whose magnitudes are B and

($1/c$) $v_0 E \sin \alpha$. If $v_0 = cB/E \sin \alpha$, the magnitudes of these vectors will be the same, and the numerator in the formula for \mathbf{B}'_{\perp} will vanish. Hence, both \mathbf{B}'_{\parallel} and \mathbf{B}'_{\perp} are absent in this case. We can show in a similar way that the field is purely magnetic with mutual perpendicularity of the fields \mathbf{B} and \mathbf{E} in the frames K' travelling in directions perpendicular to the vector \mathbf{E} at a speed v_0 equal to $cE/B \sin \alpha$ (here α is the angle between the vectors \mathbf{v}_0 and \mathbf{B}). This statement holds provided that $cE < B \sin \alpha$. If $B \leq cE$, no systems exist in which the field is purely magnetic.

63. Field Invariants

Let us form the expression

$$\mathbf{B}'^2 - \mathbf{E}'^2 = \sum B_k'^2 - \sum E_k'^2 = \sum (B_k'^2 - E_k'^2)$$

and substitute for the quantities B'_k and E'_k in it their expressions in terms of the unprimed components [see (62.2)]:

$$\begin{aligned} \mathbf{B}'^2 - \mathbf{E}'^2 &= \sum (B_k'^2 - E_k'^2) = B_x^2 - E_x^2 \\ &\quad + \frac{(B_y + \beta E_z)^2 - (E_y - \beta B_z)^2 + (B_z - \beta E_y)^2 - (E_z + \beta B_y)^2}{1 - \beta^2} \end{aligned}$$

It is a simple matter to see that the right-hand side is reduced to the form

$$(B_x^2 - E_x^2) + (B_y^2 - E_y^2) + (B_z^2 - E_z^2)$$

We thus arrive at the conclusion that the difference between the squares of the vectors \mathbf{B} and \mathbf{E} has the same value in all inertial reference frames, i.e. is an invariant:

$$\mathbf{B}^2 - \mathbf{E}^2 = \text{inv} \quad (63.1)$$

Now let us form the scalar product of the vectors \mathbf{E}' and \mathbf{B}' , i.e. the sum $\sum E'_k B'_k$. Introducing into this sum instead of E'_k and B'_k their values from (62.2), we obtain

$$\sum E'_k B'_k = E_x B_x + \frac{(E_y - \beta B_z)(B_y + \beta E_z) + (E_z + \beta B_y)(B_z - \beta E_y)}{1 - \beta^2} = \sum E_k B_k$$

Consequently, the scalar product of the vectors \mathbf{E} and \mathbf{B} is also an invariant:

$$\mathbf{E}\mathbf{B} = \text{inv} \quad (63.2)$$

Inspection of (63.2) shows that when the fields \mathbf{B} and \mathbf{E} are mutually perpendicular (i.e. $\mathbf{E}\mathbf{B} = 0$) in a certain reference frame, they are mutually perpendicular in any other inertial reference frame.

Inspection of (63.1) shows that when the magnitudes of the vectors \mathbf{B} and \mathbf{E} are the same (i.e. $B^2 - E^2 = 0$) in a certain reference frame, they are the same in any other inertial reference frame.

In addition, the following conclusions can be drawn from the invariance of expressions (63.1) and (63.2). When the vectors \mathbf{B} and \mathbf{E} form an acute (or obtuse) angle (i.e. $\mathbf{E}\mathbf{B}$ is greater or less than zero) in a certain reference frame, they form an acute (or obtuse) angle in any other frame. If $B > E$ (or $B < E$) (i.e. $B^2 - E^2$ is greater or less than zero) in a certain frame, the same relation will be retained between the magnitudes of the vectors \mathbf{B} and \mathbf{E} in any other frame.

When both invariants equal zero, the vectors \mathbf{B} and \mathbf{E} in all inertial reference frames are mutually perpendicular and equal in magnitude.

If only the invariant (63.2) equals zero, i.e. $\mathbf{E}\mathbf{B} = 0$, it is possible to find a reference frame in which either the field \mathbf{B} or \mathbf{E} is zero depending on the value of the expression $B^2 - E^2$. The opposite is also true: if one of the fields \mathbf{B} or \mathbf{E} is zero in a certain frame, the fields in any other frame will be mutually perpendicular (we already arrived at this conclusion earlier when analysing the formulas for field transformation).

One must have in view that the fields \mathbf{B} and \mathbf{E} , generally speaking, vary from point to point. Therefore, the invariants (63.1) and (63.2) may have different values at different points. The above statements on the properties of fields relate to their points for which the assumptions we have adopted are observed (for instance, the equality to zero of a given invariant, etc.). If these assumptions are observed at all the points of a field, the statements on the properties of fields will also naturally relate to all the points.

With a view to the above remarks, let us assume that at a point of a field in the frame K , the product $\mathbf{E}\mathbf{B}$ is non-zero, i.e. that the fields at the given point are not perpendicular to each other. We can now find such a reference frame K' in which the fields at a given point are parallel to each other. In this frame, $\mathbf{E}'\mathbf{B}' = E'B'$, so that we obtain two equations:

$$B'^2 - E'^2 = B^2 - E^2, \quad E'\mathbf{B}' = \mathbf{E}\mathbf{B}$$

Solving these equations simultaneously, we find the values of the quantities E' and B' in the reference frame in which the fields \mathbf{E}' and \mathbf{B}' are parallel (the vectors \mathbf{E} and \mathbf{B} are set).

The invariants of a field can be found proceeding from the general properties of tensors. It is shown in Appendix X that the multiplication of tensors of ranks m and n yields a tensor of rank $(m + n)$, and also that the contraction of a tensor over any pair of indices lowers the rank of the tensor by two. Particularly, the contraction

of a second-rank tensor, equal to the sum of its diagonal components, is called the trace of the tensor and is an invariant [see formulas (X.24) and (XII.63)].

For an antisymmetric tensor like the electromagnetic field tensor, the trace is zero so that this invariant is of no interest.

Let us form the product of the tensors (61.8) and (61.9), i.e. a tensor with the components $F_{\mu\nu}F^{\rho\sigma}$. It is a tensor of the fourth rank. We shall perform a double contraction of this tensor, assuming the indices μ and ρ , and also ν and σ , to be equal and summing over these indices. As a result, the rank of the tensor will lower by four (each of the two contractions lowers the rank by two), and we obtain a zero-rank tensor, i.e. an invariant:

$$\sum_{\mu, \nu} F_{\mu\nu}F^{\mu\nu} = \text{inv}$$

Introduction of the values of $F_{\mu\nu}$ and $F^{\mu\nu}$ [see (61.8) and (61.9)] yields

$$\sum_{\mu, \nu=0}^3 F_{\mu\nu}F^{\mu\nu} = -2 \sum_{k=1}^3 E_k^2 + 2 \sum_{k=1}^3 B_k^2 = 2(B^2 - E^2) = \text{inv} \quad (63.3)$$

which agrees with formula (63.1).

Now let us form a four-tensor of the eighth rank

$$\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\beta}F_{\gamma\delta} \quad (63.4)$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is an absolutely antisymmetric unit four-pseudotensor of the fourth rank [see Appendix XII, the text following formula (XII.68)]. The non-zero components of this tensor equal $+1$ or -1 depending on whether an even or odd number of permutations is needed to obtain the given sequence of indices μ, ν, ρ, σ from the sequence 0, 1, 2, 3. Table 63.1 gives all the permutations of the indices with an indication of the sign at 1 corresponding to them.

TABLE 63.1

| Sequence | Sign | Sequence | Sign | Sequence | Sign | Sequence | Sign |
|----------|------|----------|------|----------|------|----------|------|
| 0123 | + | 1320 | + | 2013 | + | 3210 | + |
| 0132 | - | 1302 | - | 2031 | - | 3201 | - |
| 0231 | + | 1203 | + | 2130 | + | 3102 | + |
| 0213 | - | 1230 | - | 2103 | - | 3120 | - |
| 0312 | + | 1032 | + | 2301 | + | 3021 | + |
| 0321 | - | 1023 | - | 2310 | - | 3012 | - |

The four-fold contraction of the tensor (63.4) is the invariant

$$\sum_{\mu, \nu, \rho, \sigma} \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \text{inv} \quad (63.5)$$

Having written 24 non-zero components of the sum (63.5) (the signs of the components must be taken from Table 63.1), it is not difficult to see that they can be divided into six groups each including four coinciding expressions:

$$\begin{aligned} \Sigma = & 4F_{01}F_{23} - 4F_{01}F_{32} + 4F_{02}F_{31} - 4F_{02}F_{13} \\ & + 4F_{03}F_{12} - 4F_{03}F_{21} \quad (63.6) \end{aligned}$$

Owing to the antisymmetric nature of the tensor $F_{\mu\nu}$, the products $F_{01}F_{32} = -F_{01}F_{23}$, etc. Expression (63.6) is therefore simplified as follows:

$$\Sigma = 8(F_{01}F_{23} + F_{02}F_{31} + F_{03}F_{12}) = \text{inv}$$

The expression in parentheses is obviously also invariant. Substitution of the values for $F_{\mu\nu}$ from (61.8) yields

$$E_x(-B_x) + E_y(-B_y) + E_z(-B_z) = \text{inv}$$

which coincides with (63.2).

64. Maxwell's Equations in the Four-Dimensional Form

The first pair of Maxwell's equations [i.e. Eqs. (55.12)] can be written as a single equation for the components of the tensor (61.8):

$$\frac{\partial F_{\mu\nu}}{\partial x^\rho} + \frac{\partial F_{\nu\rho}}{\partial x^\mu} + \frac{\partial F_{\rho\mu}}{\partial x^\nu} = 0 \quad (64.1)$$

We must note that the indices in each of the addends form a cyclic transposition of the sequence μ, ν, ρ .

Expression (64.1) is a set of four equations, the first of which is obtained at μ, ν, ρ equal to 0, 1, 2, respectively, the second at μ, ν, ρ equal to 1, 2, 3, the third at μ, ν, ρ equal to 2, 3, 0, and, finally, the fourth at μ, ν, ρ equal to 3, 0, 1. Owing to antisymmetry of the tensor $F_{\mu\nu}$, the equation obtained at any other combination of the three non-coinciding indices reduces to one of the four indicated equations.

Let us write Eq. (64.1) assuming that $\mu = 0, \nu = 1$, and $\rho = 2$:

$$\frac{\partial F_{01}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^0} + \frac{\partial F_{20}}{\partial x^1} = 0$$

Introducing the values of $F_{\mu\nu}$ and the coordinates x^μ , we obtain

$$\frac{\partial E_x}{\partial y} - \frac{1}{c} \frac{\partial B_z}{\partial t} - \frac{\partial E_y}{\partial x} = 0$$

whence

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c} \frac{\partial B_z}{\partial t}$$

The found relation is the z -th component of the vector equation (55.12). Similarly, the equations for $\mu = 2$, $\nu = 3$, $\rho = 0$, and $\mu = 3$, $\nu = 0$, $\rho = 1$ give the x -th and y -th components of the same equation. (An equation is obtained for the component whose index is absent in the set of values of μ , ν , ρ .)

Assuming that $\mu = 1$, $\nu = 2$, $\rho = 3$, we arrive at the formula

$$\frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} = 0$$

which after introduction of the values of $F_{\mu\nu}$ and x^μ becomes

$$-\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial z} = 0$$

This is equivalent to the second of equations (55.10).

We have thus seen that Eq. (64.1) is equivalent to the first pair of Maxwell's equations.

The second pair of Maxwell's equations, i.e. equations (55.13), can be written as

$$\sum_{\nu=0}^3 \frac{\partial F^{\mu\nu}}{\partial x^\nu} = -\frac{4\pi}{c} j^\mu \quad (\mu = 0, 1, 2, 3) \quad (64.2)$$

Indeed, let us assume, for instance, that $\mu = 1$. Equation (64.2) can therefore be written as follows:

$$\frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} = -\frac{4\pi}{c} j^1$$

($F^{11} = 0$). Substituting their values for $F^{\mu\nu}$ and x^μ , and also grouping the terms as required, we get the equation

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \frac{4\pi}{c} j_x + \frac{1}{c} \frac{\partial E_x}{\partial t}$$

which is the x -th component of the vector equation (55.13). Similarly, the equations for $\mu = 2$ and $\mu = 3$ give the y -th and z -th components of the same equation.

When $\mu = 0$, Eq. (64.2) becomes

$$\frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} = -4\pi\rho$$

Introduction of the values of $F^{\mu\nu}$ and x^μ yields

$$-\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} = -4\pi\rho$$

which coincides with the second of equations (55.13).

We have thus shown that Eq. (64.2) is equivalent to the second pair of Maxwell's equations.

65. Equation of Motion of a Particle in a Field

According to (38.16), the equation of motion of a charged particle in an electromagnetic field is

$$\frac{d}{dt} \left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} \right) = e\mathbf{E} + \frac{e}{c} [\mathbf{vB}] \quad (65.1)$$

Dividing both sides of (65.1) by $\sqrt{1-v^2/c^2}$ and taking into account that $dt\sqrt{1-v^2/c^2}$ is $d\tau$ [see formula (34.13)], we obtain

$$\left. \begin{aligned} \frac{d}{d\tau} \left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} \right) &= \frac{e}{c} \frac{c\mathbf{E} + [\mathbf{vB}]}{\sqrt{1-v^2/c^2}} \\ \text{or, in components,} \\ \frac{d}{d\tau} \left(\frac{mv_x}{\sqrt{1-v^2/c^2}} \right) &= \frac{e}{c} \frac{cE_x + v_y B_z - v_z B_y}{\sqrt{1-v^2/c^2}} \\ \frac{d}{d\tau} \left(\frac{mv_y}{\sqrt{1-v^2/c^2}} \right) &= \frac{e}{c} \frac{cE_y + v_z B_x - v_x B_z}{\sqrt{1-v^2/c^2}} \\ \frac{d}{d\tau} \left(\frac{mv_z}{\sqrt{1-v^2/c^2}} \right) &= \frac{e}{c} \frac{cE_z + v_x B_y - v_y B_x}{\sqrt{1-v^2/c^2}} \end{aligned} \right\} \quad (65.2)$$

But $v_k/\sqrt{1-v^2/c^2} = u^k$, $c/\sqrt{1-v^2/c^2} = u^0$ [see (36.6)]. Further, $E_x = F^{10}$, $E_y = F^{20}$, $E_z = F^{30}$, $B_x = F^{32} = -F^{23}$, $B_y = F^{13} = -F^{31}$, $B_z = F^{21} = -F^{12}$. Therefore, Eqs. (65.2) can be written as follows:

$$\left. \begin{aligned} m \frac{du^1}{d\tau} &= \frac{e}{c} (F^{10}u_0 + F^{12}u_2 + F^{13}u_3) \\ m \frac{du^2}{d\tau} &= \frac{e}{c} (F^{20}u_0 + F^{21}u_1 + F^{23}u_3) \\ m \frac{du^3}{d\tau} &= \frac{e}{c} (F^{30}u_0 + F^{31}u_1 + F^{32}u_2) \end{aligned} \right\} \quad (65.3)$$

On the right-hand side, we have used the covariant components of the four-velocity so that all the terms will have a plus sign.

Now let us take advantage of the fact that the rate of change in the energy of a particle equals the work done in unit time by the forces acting on the particle:

$$\frac{d}{dt} \left(\frac{mc^2}{\sqrt{1-v^2/c^2}} \right) = \left(e\mathbf{E} + \frac{e}{c} [\mathbf{vB}] \right) \mathbf{v} = e\mathbf{E}\mathbf{v} \quad (65.4)$$

Dividing both sides of this equation by $c\sqrt{1-v^2/c^2}$, we obtain

$$m \frac{du^0}{d\tau} = \frac{e}{c} (F^{01}u_1 + F^{02}u_2 + F^{03}u_3) \quad (65.5)$$

The set of equations (65.3) and (65.5) can be written as a single equation

$$m \frac{du^\mu}{d\tau} = \frac{e}{c} \sum_{\nu=0}^3 F^{\mu\nu} u_\nu \quad (\mu=0, 1, 2, 3) \quad (65.6)$$

This is the equation of motion of a particle in a field written in the four-dimensional form. Its space components are equivalent to Eq. (65.1), and its time component—to Eq. (65.4).

Equation (65.6) can be written in a somewhat different way. Let us lower the free index μ on both sides of formula (65.6). In addition, let us simultaneously lower the dummy index ν in $F^{\mu\nu}$, and raise it in u_ν . The result is

$$m \frac{du_\mu}{d\tau} = \frac{e}{c} \sum_{\nu=0}^3 F_{\mu\nu} u^\nu \quad (\mu=0, 1, 2, 3) \quad (65.7)$$

Equation (65.7) can be obtained directly from Eqs. (65.1) and (65.4) if we replace the left-hand side in them with the covariant components of the four-velocity, and write the quantities E_k and B_k as the components of the tensor (61.8).

Chapter XII

THE VARIATIONAL PRINCIPLE IN ELECTRODYNAMICS

66. Action for a Charged Particle in an Electromagnetic Field

Agreement with experiments is obtained if we take the following expression as the action for a particle in a field:

$$S = \int_1^2 \left(-mc \, ds - \frac{e}{c} \sum_{\mu=0}^3 A_{\mu} \, dx^{\mu} \right) \quad (66.1)$$

where m is the mass of the particle, e is its charge, and A_{μ} is the four-potential of the field. We draw attention to the fact that there is an invariant in the integrand, as there should be. At $A_{\mu} = 0$, expression (66.1) transforms into the action for a free particle [see formula (39.7)].

We must note that the ambiguity of the potential does not affect the equations of motion. Indeed, substituting for A_{μ} in (66.1) the quantities \bar{A}_{μ} [see formulas (60.16) and (60.17)], we get additional terms in the integrand:

$$-\frac{e}{c} \left[\sum_{k=1}^3 \frac{\partial \psi}{\partial x^k} dx^k + C \, dx^0 \right] = -\frac{e}{c} [d\psi(x^1, x^2, x^3) + C \, dx^0]$$

Integration of these terms yields a constant quantity:

$$-\frac{e}{c} \int_1^2 (d\psi + C \, dx^0) = -\frac{e}{c} [\psi(2) + Cx_{(2)}^0] + \frac{e}{c} [\psi(1) + Cx_{(1)}^0]$$

which in variation of the action vanishes.

The right-hand side of (66.1) consists of two terms. The first depends only on the properties of a particle (on m). The second describes the interaction of the particle with the field; it contains accordingly both a quantity characterizing the particle (the charge e) and a quantity characterizing the field (the potential A_{μ}). In general, we ought to include another term describing the field itself. But in considering the motion of a particle in a given field, this term may be left out of consideration because owing to the determinacy of the field it should not vary. True, this statement holds only provided that the charge of the particle is so small that we may disregard its influence on the field. In considering the motion of a particle in a given field, we shall assume that this condition is satisfied.

We shall obtain equations of motion of a particle proceeding from the principle of least action. According to this principle, the following condition is observed for the true trajectory of a particle:

$$\delta S = \delta \int_1^2 \left(-mc \sqrt{\sum dx_\mu dx^\mu} - \frac{e}{c} \sum A_\mu dx^\mu \right) = 0$$

(we have made the substitution $ds^2 = \sum dx_\mu dx^\mu$). Performing variation, we obtain

$$\delta S = \int_1^2 \left(-mc \sum \frac{dx_\mu d\delta x^\mu}{ds} - \frac{e}{c} \sum A_\mu d\delta x^\mu - \frac{e}{c} \sum \delta A_\mu dx^\mu \right) = 0 \quad (66.2)$$

[see formula (39.9); recall that $\delta dx^\mu = d\delta x^\mu$].

Let us integrate the first two terms in the integrand by parts. As a result, the first term becomes

$$\int_1^2 m \sum \delta x^\mu \frac{du_\mu}{d\tau} d\tau$$

where τ is the proper time of the particle, and u_μ is the four-velocity of the particle [see formula (39.11)]. Integration of the second term by parts yields

$$- \int_1^2 \frac{e}{c} \sum A_\mu d\delta x^\mu = - \frac{e}{c} \sum A_\mu \delta x^\mu \Big|_1^2 + \int_1^2 \frac{e}{c} \sum \delta x^\mu dA_\mu$$

The first expression on the right vanishes because at the ends of the trajectory $\delta x^\mu = 0$. Consequently, after integration by parts of the first two terms, expression (66.2) becomes

$$\delta S = \int_1^2 \left(m \sum_\mu \delta x^\mu \frac{du_\mu}{d\tau} d\tau + \frac{e}{c} \sum_\mu \delta x^\mu dA_\mu - \frac{e}{c} \sum_\nu \delta A_\nu dx^\nu \right) = 0$$

(it will come to light in the following text that it is expedient to denote the dummy index in the third term by the symbol ν).

Now let us perform the substitutions

$$dA_\mu = \sum_\nu \frac{\partial A_\mu}{\partial x^\nu} dx^\nu = \sum_\nu \frac{\partial A_\mu}{\partial x^\nu} u^\nu d\tau \text{ in the second term, and}$$

$$\delta A_\nu = \sum_\mu \frac{\partial A_\nu}{\partial x^\mu} \delta x^\mu \text{ and } dx^\nu = u^\nu d\tau \text{ in the third term}$$

The result is

$$\delta S = \int_1^2 \left(m \sum_{\mu} \delta x^{\mu} \frac{du_{\mu}}{d\tau} d\tau + \frac{e}{c} \sum_{\mu} \delta x^{\mu} \sum_{\nu} \frac{\partial A_{\mu}}{\partial x^{\nu}} u^{\nu} d\tau - \sum_{\nu} u^{\nu} d\tau \sum_{\mu} \frac{\partial A_{\nu}}{\partial x^{\mu}} \delta x^{\mu} \right) = 0$$

which can be written as

$$\delta S = \int_1^2 \sum_{\mu} \left(m \frac{du_{\mu}}{d\tau} + \frac{e}{c} \sum_{\nu} \frac{\partial A_{\mu}}{\partial x^{\nu}} u^{\nu} - \frac{e}{c} \sum_{\nu} \frac{\partial A_{\nu}}{\partial x^{\mu}} u^{\nu} \right) \delta x^{\mu} d\tau = 0$$

For the condition indicated above to be satisfied at arbitrary values of δx^{μ} , it is essential that all the terms in parentheses be zero. Hence, we obtain the relations

$$m \frac{du_{\mu}}{d\tau} = \frac{e}{c} \sum_{\nu} \left(\frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}} \right) u^{\nu}$$

or

$$m \frac{du_{\mu}}{d\tau} = \frac{e}{c} \sum_{\nu} F_{\mu\nu} u^{\nu} \quad (\mu = 0, 1, 2, 3) \quad (66.3)$$

[see formula (61.5)], which coincides with (65.7).

We have thus obtained an equation of motion of a particle in a field proceeding from the principle of least action.

67. Action for an Electromagnetic Field

In the preceding section, we considered the field in which a particle moves to be preset, and in this connection we took no account of the term in the action describing the properties of the field itself. Now let us consider a system made up of particles in an electromagnetic field and attempt to find equations determining the field proceeding from the principle of least action. Here we can give up our assumption on the smallness of the particles' charges and obtain equations for the true field, i.e. the one obtained upon superposition of the external field and that produced by the charges themselves. Consequently, the values of A_{μ} will have to depend on the positions and velocities of the particles.

The action for the system field + particle must consist of three terms:

$$S = S_f + S_m + S_{mf} \quad (67.1)$$

Here S_f is the part of the action that depends only on the properties of the field itself, i.e. the action for the field in the absence of charges; S_m is the part of the action that depends only on the properties

of the particles, i.e. the action for the free charges. And, finally, S_{mf} is the part of the action that is due to the interaction between the particles and the field.

As regards the last two terms, their values can be obtained by summing expressions (66.1) for all the particles. Hence,

$$S_{\text{m}} = - \sum_{a=1}^N m_a c \int ds_a \quad (67.2)$$

$$S_{\text{mf}} = - \sum_{a=1}^N \frac{e_a}{c} \int \sum_{\mu=0}^3 A_{\mu a} dx_a^\mu \quad (67.3)$$

Here a is the number of the particle, N is the number of particles considered, and $A_{\mu a}$ is the potential of the field at the point of four-space where the a -th particle is. The subscript a must be distinguished from the indices μ, ν, ρ, \dots used to designate the components of four-vectors and four-tensors. For the latter indices, we must distinguish between upper and lower positions. This distinction has no meaning for the index a .

Expression (67.3) can be written in a different way. For this purpose, let us substitute $(dx_a^\mu/dt) dt$ for dx_a^μ in it, i.e. write

$$S_{\text{mf}} = - \sum_a \frac{e_a}{c} \int \sum_{\mu} A_{\mu a} \frac{dx_a^\mu}{dt} dt$$

Next let us write the collection of point charges e_a as a charge distributed in space with the density

$$\rho = \sum_a e_a \delta(\mathbf{r} - \mathbf{r}_a)$$

[see (41.12)]. Therefore, the charge confined in the volume element dV can be written as $de = \rho dV$, and a sum of the form $\sum e_a f(x_a, y_a, z_a)$ can be replaced with the integral

$$\int \rho(x, y, z) f(x, y, z) dV$$

As a result, we obtain

$$S_{\text{mf}} = - \frac{1}{c} \int \rho dV \int \sum_{\mu} A_{\mu} \frac{dx^\mu}{dt} dt$$

Now let us take into account that $\rho(dx^\mu/dt) = j^\mu$, where j^μ is a component of the four-current [see formula (60.8)]. Hence, we can write

$$S_{\text{mf}} = - \frac{1}{c} \int \sum_{\mu} A_{\mu} j^\mu dV dt = - \frac{1}{c^2} \int \sum_{\mu} A_{\mu} j^\mu dV^* \quad (67.4)$$

where $dV^* = dx^0 dx^1 dx^2 dx^3 = c dV dt$.

To obtain an expression for S_t , let us take into account that, as shown experimentally, an electromagnetic field obeys the superposition principle. Therefore, the equations for a field must be linear differential equations. The equations of a field are obtained by variation of the action. But in variation, the power of the integrand lowers by one. Consequently, the equations will be linear if the action contains in the integrand an expression quadratic with respect to the field. In addition, this expression must be invariant. There are two very simple quadratic invariants that can be formed from the characteristics of a field: $\sum A_\mu A^\mu$ and $\sum F_{\mu\nu} F^{\mu\nu}$. The first of them is not suitable for our purposes because the four-potential is determined ambiguously. We thus arrive at the conclusion that the integrand in the action must consist of the invariant $\sum F_{\mu\nu} F^{\mu\nu}$. To obtain the action for an entire field, we must integrate over the entire four-space where the field is non-zero. Hence, we obtain the expression

$$S_t = \alpha \int \sum_{\mu, \nu} F_{\mu\nu} F^{\mu\nu} dV^*$$

where α is a constant; and $dV^* = c dV dt$. Integration over the coordinates is performed over the entire three-dimensional space, and over the time between two preset instants t_1 and t_2 .

We obtain correct equations of a field from the expression we have found if we assume (in the Gaussian system of units) that $\alpha = -1/16\pi c$. Hence,

$$S_t = -\frac{1}{16\pi c} \int \sum_{\mu, \nu} F_{\mu\nu} F^{\mu\nu} dV^* \quad (67.5)$$

Summation of expressions (67.2), (67.4), and (67.5) yields the action for the system field + particles:

$$S = - \sum_a m_a c \int ds_a - \frac{1}{c^2} \int \sum_\mu A_\mu j^\mu dV^* - \frac{1}{16\pi c} \int \sum_{\mu, \nu} F_{\mu\nu} F^{\mu\nu} dV^* \quad (67.6)$$

By formula (63.3), we have $\sum F_{\mu\nu} F^{\mu\nu} = 2 (\mathbf{B}^2 - \mathbf{E}^2)$. Introducing this value into (67.5), and also substituting $c dV dt$ for dV^* , we can transform the expression for S_t as follows:

$$S_t = \int_{t_1}^{t_2} dt \int \frac{1}{8\pi} (\mathbf{E}^2 - \mathbf{B}^2) dV \quad (67.7)$$

A glance at this formula shows that the Lagrangian for the field is determined by the expression

$$L_t = \frac{1}{8\pi} \int (\mathbf{E}^2 - \mathbf{B}^2) dV \quad (67.8)$$

We have thus established the form of the action for an electromagnetic field. We shall find the equations of a field proceeding from the principle of least action in the following section.

68. Derivation of Maxwell's Equations from the Principle of Least Action

Let us find the equations of a field, considering the motion of the particles to be preset. In this case, the term S_m in the action may be left out of consideration because in view of the determinacy in the motion of the charges it should not vary. We shall thus proceed from the expression

$$S = S_f + S_{mf} = -\frac{1}{16\pi c} \int \sum_{\mu, \nu} F_{\mu\nu} F^{\mu\nu} dV^* - \frac{1}{c^2} \int \sum_{\mu} A_{\mu} j^{\mu} dV^* \quad (68.1)$$

[see (67.5) and (67.4)].

Let us calculate the variation of expression (68.1) and equate it to zero. We shall take into account that owing to the determinacy of the motion of the charges the current j^{μ} should not vary. Hence,

$$\delta S = -\frac{1}{16\pi c} \int \delta \left(\sum F_{\mu\nu} F^{\mu\nu} \right) dV^* - \frac{1}{c^2} \int \sum j^{\mu} \delta A_{\mu} dV^* \quad (68.2)$$

We determine the variation in the first integrand:

$$\delta \sum F_{\mu\nu} F^{\mu\nu} = \sum F_{\mu\nu} \delta F^{\mu\nu} + \sum F^{\mu\nu} \delta F_{\mu\nu}$$

We shall raise the indices of the first factor in the first of the sums and simultaneously lower them in the second factor. The result is

$$\delta \sum F_{\mu\nu} F^{\mu\nu} = 2 \sum F^{\mu\nu} \delta F_{\mu\nu}$$

With a view to formula (61.5) for $F_{\mu\nu}$, we shall write this expression as follows:

$$\begin{aligned} \delta \sum F_{\mu\nu} F^{\mu\nu} &= 2 \sum F^{\mu\nu} \delta \left(\frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}} \right) \\ &= 2 \sum F^{\mu\nu} \delta \frac{\partial A_{\nu}}{\partial x^{\mu}} - 2 \sum F^{\mu\nu} \delta \frac{\partial A_{\mu}}{\partial x^{\nu}} \end{aligned}$$

Taking advantage of the antisymmetric nature of the tensor $F^{\mu\nu}$, we shall substitute $-F^{\nu\mu}$ for $F^{\mu\nu}$ in the first sum on the right-hand side, and then exchange the places of the indices μ and ν . As a result, the first sum will become identical to the second one, and we obtain

$$\delta \sum F_{\mu\nu} F^{\mu\nu} = -4 \sum F^{\mu\nu} \delta \frac{\partial A_{\mu}}{\partial x^{\nu}} \quad (68.3)$$

By changing the sequence of differentiation and variation of the quantities A_μ , we reduce (68.3) to the form

$$\delta \sum F_{\mu\nu} F^{\mu\nu} = -4 \sum F^{\mu\nu} \frac{\partial}{\partial x^\nu} \delta A_\mu$$

Now let us transform the expression obtained by the formula $uv' = (uv)' - u'v$:

$$\delta \sum F_{\mu\nu} F^{\mu\nu} = -4 \sum \frac{\partial}{\partial x^\nu} (F^{\mu\nu} \delta A_\mu) + 4 \sum \delta A_\mu \frac{\partial}{\partial x^\nu} F^{\mu\nu}$$

Substituting this expression into (68.2), we arrive at the following formula for the variation of the action:

$$\begin{aligned} \delta S = \frac{1}{4\pi c} \int \sum_{\mu, \nu} \frac{\partial}{\partial x^\nu} (F^{\mu\nu} \delta A_\mu) dV^* - \frac{1}{4\pi c} \int \sum_{\mu, \nu} \delta A_\mu \frac{\partial}{\partial x^\nu} F^{\mu\nu} dV^* \\ - \frac{1}{c^2} \int \sum_{\mu} j^\mu \delta A_\mu dV^* \quad (68.4) \end{aligned}$$

The first of the three integrals can be transformed according to the Ostrogradsky-Gauss formula into a surface integral:

$$\int \sum_{\mu, \nu} \frac{\partial}{\partial x^\nu} (F^{\mu\nu} \delta A_\mu) dV^* = \int \sum_{\mu, \nu} F^{\mu\nu} \delta A_\mu df_\nu$$

At the boundary of the four-volume being considered, $\delta A_\mu = 0$. Hence, the integral we have written vanishes. Consequently, only the second and third terms must be retained in formula (68.4). Combining them and factoring out the common factor δA_μ , we obtain

$$\delta S = -\frac{1}{c} \int \sum_{\mu} \left(\frac{1}{4\pi} \sum_{\nu} \frac{\partial F^{\mu\nu}}{\partial x^\nu} + \frac{1}{c} j^\mu \right) \delta A_\mu dV^*$$

Owing to the arbitrary nature of the variations δA_μ , the value of δS we have found may be zero only if all the expressions in parentheses vanish. We thus arrive at the equations

$$\sum_{\nu} \frac{\partial F^{\mu\nu}}{\partial x^\nu} = -\frac{4\pi}{c} j^\mu \quad (\mu = 0, 1, 2, 3) \quad (68.5)$$

that are the second pair of Maxwell's equations [see Eq. (64.2)].

We must note that the relation between the fields \mathbf{B} and \mathbf{E} , on the one hand, and the charges and currents, on the other, is determined exactly by the second pair of Maxwell's equations [see Eqs. (55.13)]. The first pair of Maxwell's equations expresses the properties of the fields \mathbf{B} and \mathbf{E} and their relation to each other [see Eqs. (55.12)].

69. Energy-Momentum Tensor of an Electromagnetic Field ¹

We established in Sec. 67 that the action for an electromagnetic field is determined by the expression

$$S_f = -\frac{1}{16\pi c} \int \sum_{\mu, \nu} F_{\mu\nu} F^{\mu\nu} dV^* \quad (69.1)$$

[see formula (67.5)].

A comparison of formulas (69.1) and (40.2) shows that we must take the following expression as the density of the Lagrangian for a field:

$$L^* = -\frac{1}{16\pi} \sum_{\mu, \nu} F_{\mu\nu} F^{\mu\nu} \quad (69.2)$$

We must take the components of the potential A_μ as the generalized coordinates q_a for the field, and the derivatives of these components with respect to the coordinates x^ν as the generalized velocities. To simplify the writing of the formulas, let us introduce the designation

$$\frac{\partial A_\mu}{\partial x^\nu} = a_{\mu\nu} \quad (69.3)$$

Equating the variation of the action to zero and performing the same calculations that led us to the equations of motion (40.12) in Sec. 40, we shall arrive at the relations

$$\frac{\partial L^*}{\partial A_\mu} = \sum_\nu \frac{\partial}{\partial x^\nu} \frac{\partial L^*}{\partial a_{\mu\nu}} \quad (\mu = 0, 1, 2, 3) \quad (69.4)$$

Equations (69.4) are equations of a field. To determine the values of the derivatives in them, let us write the variation of the function L^* . According to the general rules for calculating a variation, we have

$$\delta L^* = \sum_\mu \frac{\partial L^*}{\partial A_\mu} \delta A_\mu + \sum_{\mu, \nu} \frac{\partial L^*}{\partial a_{\mu\nu}} \delta a_{\mu\nu} \quad (69.5)$$

On the other hand, the variation of the function (69.2) is

$$\delta L^* = -\frac{1}{16\pi} \delta \sum_{\mu, \nu} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4\pi} \sum_{\mu, \nu} F^{\mu\nu} \delta \frac{\partial A_\mu}{\partial x^\nu} = \frac{1}{4\pi} \sum_{\mu, \nu} F^{\mu\nu} \delta a_{\mu\nu}$$

[see formula (68.3)].

A comparison of the expression obtained with formula (69.5) allows us to conclude that

$$\frac{\partial L^*}{\partial A_\mu} = 0, \quad \frac{\partial L^*}{\partial a_{\mu\nu}} = \frac{1}{4\pi} F^{\mu\nu} \quad (69.6)$$

¹ Before beginning to read this section, recall the contents of Sec. 40.

Substituting $-F^{\nu\mu}$ for $F^{\mu\nu}$ in the second expression and then exchanging the places of the indices μ and ν , we find that

$$\frac{\partial L^*}{\partial a_{\nu\mu}} = -\frac{1}{4\pi} F^{\mu\nu} \quad (69.7)$$

Introduction of the values (69.6) into formula (69.4) leads to the following "equations of motion" for an electromagnetic field:

$$\sum_{\nu} \frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = 0 \quad (\mu = 0, 1, 2, 3) \quad (69.8)$$

(we have omitted the factor $1/4\pi$). Equation (69.8) is Maxwell's equation (64.2) written for the case when $j^{\mu} = 0$. This result is what should have been expected because we proceeded from the action for only the field without any charges.

Now let us establish the form of the energy-momentum tensor for an electromagnetic field. Substituting the quantities $a_{\rho\mu}$ [see (40.3) and (69.3)] for $\dot{q}_{a\mu}$ in formula (40.16), we obtain the following expression for the components of this tensor:

$$\tilde{T}_{\mu}^{\nu} = \sum_{\rho} a_{\rho\mu} \frac{\partial L^*}{\partial a_{\rho\mu}} - \delta_{\mu}^{\nu} L^*$$

Let us replace $\partial L^* / \partial a_{\rho\nu}$ with $-(1/4\pi)F^{\nu\rho}$ in accordance with (69.7). In addition, let us substitute for $a_{\rho\mu}$ its value from (69.3), and for L^* expression (69.2). The result is

$$\tilde{T}_{\mu}^{\nu} = -\frac{1}{4\pi} \sum_{\rho} \frac{\partial A_{\rho}}{\partial x^{\mu}} F^{\nu\rho} + \frac{1}{16\pi} \delta_{\mu}^{\nu} \sum_{\beta, \gamma} F_{\beta\gamma} F^{\beta\gamma} \quad (69.9)$$

[in the second sum, we may not designate the dummy indices by the letters μ and ν as was done in formula (69.2) because in formula (69.9) μ and ν were already chosen as free indices].

The tensor (69.9) is not symmetric. To make it symmetric, let us add to it the tensor

$$G_{\mu}^{\nu} = \frac{1}{4\pi} \sum_{\rho} \frac{\partial A_{\mu}}{\partial x^{\rho}} F^{\nu\rho} \quad (69.10)$$

which, as we shall show, can be written as

$$G_{\mu}^{\nu} = \sum_{\rho} \frac{\partial Q_{\mu}^{\nu\rho}}{\partial x^{\rho}}, \quad \text{where } Q_{\mu}^{\nu\rho} = -Q_{\mu}^{\rho\nu}$$

[see formula (40.18)]. Indeed, let us apply to expression (69.10) the transformation $uv' = (uv)' - u'v$:

$$G_{\mu}^{\nu} = \frac{1}{4\pi} \sum_{\rho} \frac{\partial A_{\mu}}{\partial x^{\rho}} F^{\nu\rho} = \frac{1}{4\pi} \sum_{\rho} \frac{\partial}{\partial x^{\rho}} (A_{\mu} F^{\nu\rho}) - \frac{1}{4\pi} \sum_{\rho} A_{\mu} \frac{\partial F^{\nu\rho}}{\partial x^{\rho}}$$

Owing to (69.8) the second sum on the right-hand side vanishes (A_μ can be factored out of the sum sign). Consequently, we have brought the tensor (69.10) to the form

$$G_\mu^\nu = \sum_\rho \frac{\partial}{\partial x^\rho} \left(\frac{1}{4\pi} A_\mu F^{\nu\rho} \right)$$

Since the expression in parentheses is antisymmetric with respect to the indices ν and ρ , expression (69.10) may be added to the tensor (69.9) [see formula (40.18) and the text associated with it].

Summation of expressions (69.9) and (69.10) gives the tensor

$$\begin{aligned} T_\mu^\nu &= -\frac{1}{4\pi} \sum_\rho \frac{\partial A_\rho}{\partial x^\mu} F^{\nu\rho} + \frac{1}{4\pi} \sum_\rho \frac{\partial A_\mu}{\partial x^\rho} F^{\nu\rho} + \frac{1}{16\pi} \delta_\mu^\nu \sum_{\beta, \gamma} F_{\beta\gamma} F^{\beta\gamma} \\ &= -\frac{1}{4\pi} \sum_\rho \left(\frac{\partial A_\rho}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\rho} \right) F^{\nu\rho} + \frac{1}{16\pi} \delta_\mu^\nu \sum_{\beta, \gamma} F_{\beta\gamma} F^{\beta\gamma} \end{aligned}$$

The expression in parentheses is $F_{\mu\rho}$ [see (61.5)]. Consequently, the formula for the mixed components of the energy-momentum tensor becomes

$$T_\mu^\nu = -\frac{1}{4\pi} \sum_\rho F_{\mu\rho} F^{\nu\rho} + \frac{1}{16\pi} \delta_\mu^\nu \sum_{\beta, \gamma} F_{\beta\gamma} F^{\beta\gamma} \quad (69.11)$$

To go over to contravariant components, we shall raise the index μ in all the terms of formula (69.11). Here δ_μ^ν will transform into $g^{\mu\nu}$ [see (XII.66) and (XII.67)]. Hence,

$$T^{\mu\nu} = -\frac{1}{4\pi} \sum_\rho F^\mu{}_\rho F^{\nu\rho} + \frac{1}{16\pi} g^{\mu\nu} \sum_{\beta, \gamma} F_{\beta\gamma} F^{\beta\gamma} \quad (69.12)$$

Raising of an index in one factor with simultaneous lowering of the same index in the second factor does not change the product. We can therefore write with equal grounds that

$$T^{\mu\nu} = -\frac{1}{4\pi} \sum_\rho F^{\mu\rho} F^\nu{}_\rho + \frac{1}{16\pi} g^{\mu\nu} \sum_{\beta, \gamma} F_{\beta\gamma} F^{\beta\gamma} \quad (69.13)$$

A comparison of expressions (62.12) and (62.13) allows us to conclude that the tensor $T^{\mu\nu}$ is indeed symmetric.

Let us calculate the trace of the tensor $T^{\mu\nu}$. According to formula (69.11),

$$\sum_\mu T_\mu^\mu = \sum_\mu \left(-\frac{1}{4\pi} \sum_\rho F_{\mu\rho} F^{\mu\rho} \right) + \sum_\mu \left(\frac{1}{16\pi} \delta_\mu^\mu \sum_{\beta, \gamma} F_{\beta\gamma} F^{\beta\gamma} \right)$$

In the second term, all the factors except δ_μ^μ can be put outside the sign of summation over μ . The sum $\sum \delta_\mu^\mu$ equals four. Hence

$$\sum_{\mu} T_\mu^\mu = -\frac{1}{4\pi} \sum_{\mu, \rho} F_{\mu\rho} F^{\mu\rho} + 4 \frac{1}{16\pi} \sum_{\beta, \gamma} F_{\beta\gamma} F^{\beta\gamma}$$

It is not difficult to see that the last expression vanishes. We have therefore established that the trace of the tensor $T^{\mu\nu}$ is zero:

$$\sum_{\mu} T_\mu^\mu = 0 \quad (69.14)$$

Let us find the expressions for the components of the tensor $T^{\mu\nu}$ in terms of the components of the vectors \mathbf{E} and \mathbf{B} . For this purpose, we shall introduce the values of the components $F^{\mu\nu}$ into (69.13). We shall take these values from (61.9), having in mind that lowering of the time index does not change the components of the tensor, while lowering of the space index reverses the sign of a component. We shall also take into account that the sum $\sum F_{\mu\nu} F^{\mu\nu}$ is an invariant that we calculated in Sec. 63. It equals

$$\sum_{\mu, \nu} F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2) \quad (69.15)$$

[see formula (63.3)].

Let us begin with calculation of T^{00} . According to (69.13), (61.9), and (69.15), we have

$$\begin{aligned} T^{00} &= -\frac{1}{4\pi} \sum_{\rho} F^{0\rho} F_\rho^0 + \frac{1}{16\pi} g^{00} 2(B^2 - E^2) \\ &= -\frac{1}{4\pi} (-E^2) + \frac{1}{16\pi} 2(B^2 - E^2) = \frac{1}{8\pi} (E^2 + B^2) = w \end{aligned}$$

[$g^{00} = 1$; see (XII.67)]. We have obtained an already known result: the component T^{00} equals the energy density w [see formula (40.27)].

Now let us find the component T^{01} . Since $g^{01} = 0$,

$$\begin{aligned} T^{01} &= -\frac{1}{4\pi} \sum_{\rho} F^{0\rho} F_\rho^1 \\ &= -\frac{1}{4\pi} [0 \cdot E_x + (-E_x) \cdot 0 + (-E_y) B_z + (-E_z)(-B_y)] \\ &= \frac{1}{4\pi} (E_y B_z - E_z B_y) = \left\{ \frac{1}{4\pi} [\mathbf{E} \mathbf{B}] \right\}_x = \frac{1}{c} S_x \quad (69.16) \end{aligned}$$

where S_x is the x -th component of the Poynting vector (in a vacuum $\mathbf{H} = \mathbf{B}$). Similar calculations show that $T^{02} = S_y/c$, and $T^{03} = S_z/c$ [compare with formula (40.32)].

Finally, let us calculate, for example, T^{12} . According to (XII.67), $g^{12} = 0$. Therefore

$$\begin{aligned} T^{12} &= -\frac{1}{4\pi} \sum_{\rho} F^{1\rho} F^2_{\rho} \\ &= -\frac{1}{4\pi} [E_x E_y + 0 \cdot (-B_z) + (-B_z) \cdot 0 + B_y B_x] \\ &= -\frac{1}{4\pi} (E_x E_y + B_x B_y) = \sigma_{xy} \end{aligned}$$

where σ_{xy} is a component of the Maxwell stress tensor [see formula (59.12)]. It is a simple matter to verify that all the remaining components having the form T^{ih} coincide with the relevant components of the tensor (59.12):

$$T^{ih} = \sigma_{ih} = \frac{1}{4\pi} \left\{ \frac{1}{2} (E^2 + B^2) \delta_{ih} - E_i E_h - B_i B_h \right\} \quad (69.17)$$

[in verifying, it must be taken into account that $g^{ii} = -1 = -\delta_{ii}$; see (XII.67)].

Let us consider the matter of diagonalization of the tensor $T^{\mu\nu}$. In Euclidean space, such a transformation of a symmetric tensor is always possible. In pseudo-Euclidean space, however, matters are different, as we shall now see. According to (69.16), components of the form $T^{0i} = T^{i0}$ vanish provided that

$$[\mathbf{E}\mathbf{B}] = 0 \quad (69.18)$$

According to (69.17), components of the form T^{ik} ($i \neq k$) vanish provided that

$$E_i E_k = 0 \text{ and } B_i B_k = 0 \quad (i \neq k) \quad (69.19)$$

Therefore, for diagonalization of the tensor $T^{\mu\nu}$, we must transfer to a reference frame in which the vectors \mathbf{B} and \mathbf{E} are collinear, or one of them vanishes [in which case condition (69.18) is satisfied]. We established in Sec. 62 that such a frame always exists except when \mathbf{B} and \mathbf{E} are mutually perpendicular and identical in magnitude. In this reference frame, one of the coordinate axes must be directed along the field. This will result in condition (69.19) being satisfied. The tensor will therefore acquire a diagonal form. Let us find the components T^{ii} assuming that the x -axis has been chosen in the direction of the fields and, consequently,

$$E_x = E, \quad E_y = E_z = 0, \quad B_x = \pm B, \quad B_y = B_z = 0$$

Hence, by (69.17),

$$\begin{aligned} T^{11} &= \frac{1}{4\pi} \left\{ \frac{1}{2} (E^2 + B^2) - E^2 - B^2 \right\} = -\frac{1}{8\pi} (E^2 + B^2) = -w \\ T^{22} &= T^{33} = \frac{1}{8\pi} (E^2 + B^2) = w \end{aligned}$$

We know that the component T^{00} also equals w . Being reduced to the diagonal form, the energy-momentum tensor of an electromagnetic field is as follows:

$$(T^{\mu\nu}) = \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & -w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix} \quad (69.20)$$

If the vectors \mathbf{B} and \mathbf{E} are mutually perpendicular and identical in magnitude, the tensor $T^{\mu\nu}$ cannot be given a diagonal form (the mutual perpendicularity does not allow \mathbf{B} and \mathbf{E} to be transformed so that they become collinear, and the equality of their magnitudes does not allow them to be transformed so that one of the fields vanishes).

We must note that in mixed components, the tensor (69.20) is

$$(T^\mu_\nu) = \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & -w & 0 \\ 0 & 0 & 0 & -w \end{pmatrix} \quad (69.21)$$

It directly follows from (69.21) that the trace of the tensor $T^{\mu\nu}$ is zero. We obtained this result earlier in the general case [see (69.14)].

70. A Charged Particle in an Electromagnetic Field

We know that the action for a charged particle in an electromagnetic field is determined by the expression

$$S = \int_1^2 \left(-mc \, ds - \frac{e}{c} \sum_{\mu=0}^3 A_\mu \, dx^\mu \right) \quad (70.1)$$

[see formula (66.1)].

The covariant components of the four-potential can be written as

$$A_\mu = (\varphi, -\mathbf{A})$$

[see (60.15)]. The components of the four-position vector are

$$x^\mu = (ct, \mathbf{r})$$

Consequently,

$$\sum_{\mu=0}^3 A_\mu \, dx^\mu = \varphi c \, dt - \mathbf{A} \, d\mathbf{r} = (c\varphi - \mathbf{A}\mathbf{v}) \, dt$$

Let us substitute this value of the sum into formula (70.1). In addition, let us substitute $c \sqrt{1 - v^2/c^2} \, dt$ for ds in accordance

with (34.4). The result is

$$S = \int_1^2 \left(-mc^2 \sqrt{1 - v^2/c^2} + \frac{e}{c} \mathbf{A} \mathbf{v} - e\varphi \right) dt$$

Hence we conclude that the Lagrangian for a charged particle in a field is

$$L = -mc^2 \sqrt{1 - v^2/c^2} + \frac{e}{c} \mathbf{A} \mathbf{v} - e\varphi \quad (70.2)$$

The first term is the Lagrangian for a free particle [see (39.4)]. The other two terms describe the interaction of the particle with the field.

Knowing the Lagrangian, we can calculate the energy and momentum of a particle. According to formulas (4.19) and (5.1), the generalized momentum is determined by the expression

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{v}}$$

and the energy, by the expression

$$W = \frac{\partial L}{\partial \mathbf{v}} \mathbf{v} - L \quad (70.3)$$

Consequently, having differentiated the function (70.2) with respect to \mathbf{v} , we obtain the generalized momentum of a particle:

$$\mathbf{P} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} + \frac{e}{c} \mathbf{A} = \mathbf{p} + \frac{e}{c} \mathbf{A} \quad (70.4)$$

Here \mathbf{p} is the conventional momentum of the particle [see formula (38.5)]. A glance at formula (70.4) shows that the generalized momentum differs from the conventional one in the term $(e/c) \mathbf{A}$. In the absence of a field, the generalized momentum coincides with the conventional one.

Now let us determine the energy of a particle. In accordance with (70.3) and (70.2), we have

$$\begin{aligned} W &= \frac{\partial L}{\partial \mathbf{v}} \mathbf{v} - L = \mathbf{P} \mathbf{v} - L \\ &= \left(\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} + \frac{e}{c} \mathbf{A} \right) \mathbf{v} - \left(-mc^2 \sqrt{1 - v^2/c^2} + \frac{e}{c} \mathbf{A} \mathbf{v} - e\varphi \right) \\ &= \frac{mc^2}{\sqrt{1 - v^2/c^2}} + e\varphi \quad (70.5) \end{aligned}$$

The first term in (70.5) is the energy of a free particle [see formula (38.11)], the second term is the additional energy that a particle in the field has.

Replacing the velocity \mathbf{v} in the expression for the energy in terms of the generalized momentum \mathbf{P} , let us find the Hamiltonian of a particle. To delete \mathbf{v} from formulas (70.5) and (70.4), let us write these equations as

$$\frac{W - e\varphi}{c} = \frac{mc}{\sqrt{1 - v^2/c^2}}$$

$$\mathbf{P} - \frac{e}{c} \mathbf{A} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}$$

If we square these equations and subtract the lower one from the upper one, on the right we obtain m^2c^2 . Consequently, substituting \mathcal{H} for W , we have

$$\left(\frac{\mathcal{H} - e\varphi}{c}\right)^2 - \left(\mathbf{P} - \frac{e}{c} \mathbf{A}\right)^2 = m^2c^2 \quad (70.6)$$

whence

$$\mathcal{H} = c \sqrt{m^2c^2 + \left(\mathbf{P} - \frac{e}{c} \mathbf{A}\right)^2} + e\varphi \quad (70.7)$$

This is the Hamiltonian for a particle in a field.

It was established in Sec. 32 that the components of the generalized momentum equal the derivatives of the action with respect to the relevant generalized coordinates [see formula (32.6)]. In our case, the role of the generalized coordinates is played by the Cartesian coordinates x_i . Consequently,

$$P_i = \frac{\partial S}{\partial x_i} \quad \text{or} \quad \mathbf{P} = \nabla S$$

Further, according to (32.10), the derivative of the action with respect to time gives the Hamiltonian taken with the opposite sign:

$$\frac{\partial S}{\partial t} = -\mathcal{H}$$

Substituting $-\partial S/\partial t$ for \mathcal{H} and ∇S for \mathbf{P} in formula (70.6), we arrive at the Hamilton-Jacobi equation for a particle in an electromagnetic field:

$$\frac{1}{c^2} \left(\frac{\partial S}{\partial t} + e\varphi \right)^2 - \left(\nabla S - \frac{e}{c} \mathbf{A} \right)^2 - m^2c^2 = 0 \quad (70.8)$$

In the classical approximation, i.e. when $v \ll c$, the function (70.2) becomes

$$L = \frac{mv^2}{2} + \frac{e}{c} \mathbf{A} \mathbf{v} - e\varphi \quad (70.9)$$

[we have expanded (70.2) by the powers of v^2/c^2 and discarded the constant $-mc^2$].

Differentiation of (70.9) with respect to \mathbf{v} yields the generalized momentum

$$\mathbf{P} = m\mathbf{v} + \frac{e}{c} \mathbf{A} = \mathbf{p} + \frac{e}{c} \mathbf{A} \quad (70.10)$$

where \mathbf{p} is the conventional momentum.

For the energy in this approximation, we obtain

$$W = \mathbf{P}\mathbf{v} - L = \left(m\mathbf{v} + \frac{e}{c} \mathbf{A} \right) \mathbf{v} - \frac{mv^2}{2} - \frac{e}{c} \mathbf{A}\mathbf{v} + e\varphi = \frac{mv^2}{2} + e\varphi \quad (70.11)$$

From (70.10), we have

$$\mathbf{v} = \frac{1}{m} \left(\mathbf{P} - \frac{e}{c} \mathbf{A} \right)$$

Introducing this value of \mathbf{v} into (70.11), we arrive at an expression for the Hamiltonian:

$$\mathcal{H} = \frac{1}{2m} \left(\mathbf{P} - \frac{e}{c} \mathbf{A} \right)^2 + e\varphi \quad (70.12)$$

The Hamilton-Jacobi equation in the classical approximation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\nabla S - \frac{e}{c} \mathbf{A} \right)^2 + e\varphi = 0 \quad (70.13)$$

71. The Wave Equation

It was shown in Sec. 57 that upon application to the potentials \mathbf{A} and φ of the Lorentz condition

$$\nabla \mathbf{A} + \frac{\varepsilon \mu}{c} \frac{\partial \varphi}{\partial t} = 0 \quad (71.1)$$

[see formula (56.8)] they satisfy d'Alembert's equation:

$$\square \mathbf{A} = -\frac{4\pi\mu}{c} \mathbf{j} \quad (71.2)$$

$$\square \varphi = -\frac{4\pi\rho}{\varepsilon} \quad (71.3)$$

[see Eqs.¹ (57.6) and (57.7)]. Here \square is d'Alembert's operator equal to

$$\square = \Delta - \frac{\varepsilon \mu}{c^2} \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\varepsilon \mu}{c^2} \frac{\partial^2}{\partial t^2} \quad (71.4)$$

[see (57.4)].

In the absence of charges and currents (i.e. at $\rho = 0$ and $\mathbf{j} = 0$), the equations for the potentials acquire the form

$$\square \mathbf{A} = 0 \quad (71.5)$$

$$\square \varphi = 0 \quad (71.6)$$

or, with account of (71.4)

$$\Delta \mathbf{A} - \frac{\varepsilon \mu}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \quad (71.7)$$

$$\Delta \varphi - \frac{\varepsilon \mu}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \quad (71.8)$$

Similar equations are also obtained for the vectors \mathbf{E} and \mathbf{B} :

$$\Delta \mathbf{E} - \frac{\varepsilon \mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad (71.9)$$

$$\Delta \mathbf{B} - \frac{\varepsilon \mu}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \quad (71.10)$$

(see Sec. 74).

¹ Recall that Eqs. (57.6) and (57.7) were obtained on the assumption that the medium in which the field is being considered is homogeneous and isotropic, and, in addition, that ε and μ do not depend on \mathbf{E} and \mathbf{H} .

Equations (71.7)-(71.10) have non-zero solutions. Consequently, electromagnetic fields can also exist in the absence of charges. Such electromagnetic fields are known as electromagnetic waves.

An equation of the kind

$$\Delta f - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0 \quad (71.11)$$

where v is a constant, is called a wave equation. It is known from the general course of physics that v is the phase velocity of a wave. Consequently, the velocity of electromagnetic waves is

$$v = \frac{c}{\sqrt{\epsilon\mu}} = \frac{c}{n} \quad (71.12)$$

where

$$n = \sqrt{\epsilon\mu} \quad (71.13)$$

is the refractive index of the medium in which a wave propagates.

It is a simple matter to obtain the wave equation in the four-dimensional form. Let us do this for the field in a vacuum. By (64.2), Maxwell's equations in the absence of charges and currents are

$$\sum_{\nu} \frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = 0 \quad (\mu = 0, 1, 2, 3) \quad (71.14)$$

The substitution for $F^{\mu\nu}$ of their values from (61.6) yields

$$\sum_{\nu} \frac{\partial}{\partial x^{\nu}} \left(\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} \right) = 0$$

or

$$\sum_{\nu} \frac{\partial^2 A^{\nu}}{\partial x_{\mu} \partial x^{\nu}} - \sum_{\nu} \frac{\partial^2 A^{\mu}}{\partial x_{\nu} \partial x^{\nu}} = 0 \quad (71.15)$$

The first sum can be written as follows:

$$\sum_{\nu} \frac{\partial^2 A^{\nu}}{\partial x_{\mu} \partial x^{\nu}} = \frac{\partial}{\partial x_{\mu}} \sum_{\nu} \frac{\partial A^{\nu}}{\partial x^{\nu}}$$

If the four-potential satisfies the Lorentz condition (60.18), then $\sum (\partial A^{\nu} / \partial x^{\nu}) = 0$, and the first term in Eq. (71.15) will vanish. We thus arrive at the equations

$$\sum_{\nu} \frac{\partial^2 A^{\mu}}{\partial x_{\nu} \partial x^{\nu}} = 0 \quad (\mu = 0, 1, 2, 3) \quad (71.16)$$

Raising of the index on ∂x_{ν} is equivalent to multiplying each of the addends in (71.16) by $g^{\nu\nu}$ [see (XII.30)]. Therefore, Eq. (71.16) can be given the form

$$\sum_{\nu} g^{\nu\nu} \frac{\partial^2 A^{\mu}}{\partial x^{\nu} \partial x^{\nu}} = 0$$

Finally, taking into account that the non-diagonal components of the tensor $g^{\nu\rho}$ are zeros, we can write

$$\sum_{\nu, \rho} g^{\nu\rho} \frac{\partial^2 A^\mu}{\partial x^\nu \partial x^\rho} = 0 \quad (\mu = 0, 1, 2, 3) \quad (71.17)$$

Equations (71.16) and (71.17) are wave equations in the four-dimensional form. Substitution of values for $g^{\nu\rho}$ and x^ν in them results, as can readily be seen, in Eqs. (71.7) and (71.8).

72. A Plane Electromagnetic Wave in a Homogeneous and Isotropic Medium

The solution of the wave equation is considerably facilitated if the field depends on only one coordinate, say x . The wave in this case is called **plane**. Using a plane wave as an example, we can determine all the features of electromagnetic waves. For these reasons, we shall treat only plane waves in this chapter.

Let f stand for any of the components of the vector potential \mathbf{A} or the scalar potential φ (with equal reason, we can understand f to signify any component of the vector \mathbf{E} or the vector \mathbf{B}). For a plane wave, the function f depends only on x and t and, consequently, is a solution of the equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 f}{\partial t^2} = 0 \quad (72.1)$$

This equation can be written as

$$Df = 0 \quad (72.2)$$

where D is a differential operator determined by the formula

$$D = \frac{\partial^2}{\partial x^2} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2}{\partial t^2}$$

We shall represent this operator as

$$D = \left(\frac{\partial}{\partial x} - \frac{n}{c} \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial x} + \frac{n}{c} \frac{\partial}{\partial t} \right) \quad (72.3)$$

where $n = \sqrt{\varepsilon\mu}$ [see (71.13)].

Let us introduce the new variables

$$\xi = t - \frac{x}{c/n}, \quad \eta = t + \frac{x}{c/n} \quad (72.4)$$

i.e. replace x and t by the formulas

$$x = \frac{\eta - \xi}{2} \frac{c}{n}, \quad t = \frac{\eta + \xi}{2} \quad (72.5)$$

By (72.5), we have

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial t} \frac{\partial t}{\partial \xi} = -\frac{c}{2n} \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial}{\partial t} = -\frac{c}{2n} \left(\frac{\partial}{\partial x} - \frac{n}{c} \frac{\partial}{\partial t} \right)$$

Consequently, in going over to the variables ξ and η , the first factor in (72.3) must be replaced with $-(2n/c) \partial/\partial \xi$.

Similarly,

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial}{\partial t} \frac{\partial t}{\partial \eta} = \frac{c}{2n} \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial}{\partial t} = \frac{c}{2n} \left(\frac{\partial}{\partial x} + \frac{n}{c} \frac{\partial}{\partial t} \right)$$

so that the second factor in (72.3) must be replaced with $(2n/c) \partial/\partial \eta$. The result is

$$D = -\frac{4n^2}{c^2} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} = -\frac{4n^2}{c^2} \frac{\partial^2}{\partial \xi \partial \eta}$$

Let us introduce the found value of D into Eq. (72.2), discarding the factor $-4n^2/c^2$. The result is the differential equation

$$\frac{\partial^2 f}{\partial \xi \partial \eta} = 0 \quad (72.6)$$

An obvious solution of this equation is a function depending on only one variable ξ or η , i.e. the function $f_1(\xi)$ or $f_2(\eta)$. Summating the functions f_1 and f_2 , we get a general solution of Eq. (72.6):

$$f(\xi, \eta) = f_1(\xi) + f_2(\eta) \quad (72.7)$$

We did not multiply f_1 by C_1 and f_2 by C_2 because f_1 and f_2 are arbitrary functions of the relevant variables.

Introducing into (72.7) expressions (72.4) for ξ and η , we arrive at a solution of Eq. (72.1):

$$f(x, t) = f_1\left(t - \frac{x}{c/n}\right) + f_2\left(t + \frac{x}{c/n}\right) \quad (72.8)$$

The first term in this expression is a wave running at the velocity c/n in the direction of the x -axis. Indeed, the values of f_1 are the same for all the values of t and x related by the expression

$$t - \frac{x}{c/n} = \text{const} \text{ or } x = \frac{c}{n} t - \frac{c}{n} \text{const}$$

whence it follows that any preset value of the function f_1 travels along the x -axis at the velocity c/n .

Similarly, the second term in expression (72.8) is a wave travelling at the velocity c/n in a direction opposite to the x -axis.

The shape of the wave (72.8), i.e. the form of the functions f_1 and f_2 , is absolutely arbitrary.

Consider a plane wave propagating in the direction of the x -axis. Let us choose the potentials so that

$$\varphi = 0, \quad \nabla \mathbf{A} = 0 \quad (72.9)$$

Such a gauging of the potentials satisfies the Lorentz condition (71.1) and, consequently, does not affect the values of the vectors \mathbf{B} and \mathbf{E} (see Sec. 56). When we apply condition (72.9) to the potential φ , we obtain

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \quad (72.10)$$

[see (56.3)].

For the wave being considered, all the quantities characterizing a field including \mathbf{A} have the form

$$\mathbf{A} = A(\xi) = A\left(t - \frac{x}{c/n}\right) \quad (72.11)$$

so that

$$\frac{\partial A_i}{\partial y} = 0 \text{ and } \frac{\partial A_i}{\partial z} = 0 \quad (i = x, y, z)$$

Particularly, $\partial A_y / \partial y$ and $\partial A_z / \partial z$ vanish. Hence, the second of conditions (72.9) in the case being considered becomes

$$\frac{\partial A_x}{\partial x} = 0$$

Introduction of the last relation into Eq. (72.1) written for A_x yields

$$\frac{\partial^2 A_x}{\partial t^2} = 0$$

According to the last equation

$$\frac{\partial A_x}{\partial t} = \text{const} \quad (72.12)$$

The derivative $\partial A_x / \partial t$ determines the field component E_x [see (72.10)]. Relation (72.12) therefore signifies that $E_x = \text{const}$. We have thus arrived at the conclusion that a non-zero component E_x can be due only to a constant and homogeneous electric field. Such a field has no relation to an electromagnetic wave. Consequently, we can consider that A_x and, therefore, E_x too, vanishes.

Examination of (72.11) shows that

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial \mathbf{A}}{\partial \xi}, \quad \frac{\partial \mathbf{A}}{\partial x} = \frac{\partial \mathbf{A}}{\partial \xi} \frac{\partial \xi}{\partial x} = -\frac{n}{c} \frac{\partial \mathbf{A}}{\partial \xi} \quad (72.13)$$

According to the first of these formulas

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial \xi} \quad (72.14)$$

Since \mathbf{A} depends only on the single coordinate x , in the expression $\mathbf{B} = [\nabla \mathbf{A}]$ we must retain only the component ∇ equal to

$\mathbf{e}_x (\partial/\partial x)$. We can therefore write that

$$\mathbf{B} = \left[\mathbf{e}_x \frac{\partial}{\partial x}, \mathbf{A} \right] = \left[\mathbf{e}_x, \frac{\partial \mathbf{A}}{\partial x} \right]$$

or, according to the second of formulas (72.13),

$$\mathbf{B} = \left[\mathbf{e}_x, \left(-\frac{n}{c} \frac{\partial \mathbf{A}}{\partial \xi} \right) \right]$$

Finally, taking into account (72.14), we obtain

$$\mathbf{B} = n [\mathbf{e}_x, \mathbf{E}] = \sqrt{\varepsilon\mu} [\mathbf{e}_x, \mathbf{E}] \quad (72.15)$$

whence it follows that the vector \mathbf{B} is perpendicular both to the vector \mathbf{E} and to the x -axis. We showed above that for a wave $E_x = 0$ and, consequently, the vector \mathbf{E} is perpendicular to the x -axis. We thus arrive at the conclusion that in a plane wave the vectors \mathbf{E} and \mathbf{B} are perpendicular to the direction of propagation of the wave. Consequently, electromagnetic waves are transverse.

In formula (72.15), \mathbf{e}_x is the unit vector of the direction in which the wave propagates. Designating this unit vector by the symbol \mathbf{k}_0 , we arrive at the formula

$$\mathbf{B} = \sqrt{\varepsilon\mu} [\mathbf{k}_0, \mathbf{E}] \quad (72.16)$$

that does not depend on our choice of the directions of the coordinate axes.

Substitution of $\mu\mathbf{H}$ for \mathbf{B} yields

$$\sqrt{\mu}\mathbf{H} = [\mathbf{k}_0, \sqrt{\varepsilon}\mathbf{E}] \quad (72.17)$$

It can be seen from (72.17) that the vectors \mathbf{H} and \mathbf{E} are mutually perpendicular, and their magnitudes are related by the expression

$$\sqrt{\mu}H = \sqrt{\varepsilon}E \quad (72.18)$$

In addition, it is easy to conclude from (72.17) that the vectors \mathbf{E} , \mathbf{H} , and \mathbf{k}_0 form a right-handed sequence.

An electromagnetic wave carries an energy whose flux is determined by the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi} [\mathbf{E}\mathbf{H}]$$

Let us express \mathbf{H} in terms of \mathbf{E} in accordance with (72.17) and use formula (VI.5). Hence

$$\mathbf{S} = \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} [\mathbf{E}, [\mathbf{k}_0\mathbf{E}]] = \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} E^2 \mathbf{k}_0 - \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} (E\mathbf{k}_0) \mathbf{E}$$

Since the vectors \mathbf{E} and \mathbf{k}_0 are mutually perpendicular, $E\mathbf{k}_0 = 0$ so that

$$\mathbf{S} = \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} E^2 \mathbf{k}_0 = \frac{c}{4\pi \sqrt{\varepsilon\mu}} \varepsilon E^2 \mathbf{k}_0$$

Using relation (72.18), we can write

$$S = \frac{c}{4\pi\sqrt{\epsilon\mu}} \epsilon E^2 \mathbf{k}_0 = \frac{c}{4\pi\sqrt{\epsilon\mu}} \mu H^2 \mathbf{k}_0 = \frac{c}{\sqrt{\epsilon\mu}} \frac{(\epsilon E^2 + \mu H^2)}{8\pi} \mathbf{k}_0$$

Finally, with a view to $(\epsilon E^2 + \mu H^2)/8\pi$ being the wave energy density w , we obtain

$$S = \frac{c}{\sqrt{\epsilon\mu}} w \mathbf{k}_0 = v w \mathbf{k}_0 \quad (72.19)$$

In a vacuum

$$S = c w \mathbf{k}_0 \quad (72.20)$$

Expression (72.19) agrees with the relation known from the general course of physics, according to which the density of the energy flux carried by a wave equals the density of the energy multiplied by the velocity of propagation of the wave. The direction of the energy flux density vector coincides with that of wave propagation. The momentum flux is determined by Maxwell's stress tensor whose components are calculated by the formula

$$\sigma_{ik} = \frac{1}{4\pi} \left\{ E_i D_k + H_i B_k - \frac{1}{2} (ED + HB) \delta_{ik} \right\}$$

[see (59.12)]. Let us direct the x -axis along \mathbf{k}_0 , the y -axis along \mathbf{E} , and the z -axis along \mathbf{H} (the vectors \mathbf{E} , \mathbf{H} , and \mathbf{k}_0 will form a right-handed sequence). Consequently, $E_y = \pm E$, $E_x = E_z = 0$, $D_y = \pm D$, $D_x = D_z = 0$, $H_z = \pm H$, $H_x = H_y = 0$, $B_z = \pm B$, $B_x = B_y = 0$. It can be seen that with such a choice of the axes all the non-diagonal components of the tensor σ_{ik} vanish. Since $E_x D_x$ and $H_x B_x$ equal zero, we obtain that

$$\sigma_{xx} = -\frac{ED + HB}{8\pi} = -w$$

Since $E_y D_y = ED$, and $H_y B_y = 0$, we have

$$\sigma_{yy} = \frac{1}{4\pi} \left\{ ED - \frac{1}{2} (ED + HB) \right\} = 0$$

[by (72.18), $ED = HB$]. Similarly

$$\sigma_{zz} = \frac{1}{4\pi} \left\{ HB - \frac{1}{2} (ED + HB) \right\} = 0$$

Hence, with our choice of the coordinate axes, only one component of Maxwell's stress tensor is non-zero—the component σ_{xx} , and it equals the wave energy density w taken with the opposite sign. We remind our reader that in accordance with the customary practice in the overwhelming majority of manuals on electrodynamics, we defined the tensor σ_{ik} to characterize the flux of the momentum not flowing out of a given volume, but flowing into it (see Sec. 59). This is equivalent to reversing the direction of a normal to an area.

Hence, the momentum flux carried in the direction of the x -axis through the area df perpendicular to this axis is determined by a positive quantity, namely, $\sigma_{xx}(-df) = w df$.

73. A Monochromatic Plane Wave

A wave in which the field at each point varies with time according to a harmonic law (i.e. according to a cosine law) is called **monochromatic**.

For a monochromatic wave, the solution of Eq. (71.11) is

$$f = a \cos(\omega t + \psi_1) \quad (73.1)$$

where a is a quantity not depending on t , and ψ_1 is a function of r . In this case, $\partial^2 f / \partial t^2 = -\omega^2 f$. Substitution of this value into (71.11) yields the equation

$$\Delta f + \frac{\omega^2}{v^2} f = 0 \quad (73.2)$$

that determines how f depends on r .

Let us introduce the quantity

$$\mathbf{k} = \frac{2\pi}{\lambda} \mathbf{k}_0 = \frac{\omega}{v} \mathbf{k}_0 = \sqrt{\epsilon\mu} \frac{\omega}{c} \mathbf{k}_0 \quad (73.3)$$

known as the **wave vector** (\mathbf{k}_0 is the unit vector of the direction in which the wave is propagating). Equation (73.2) can now be written as follows:

$$\Delta f + k^2 f = 0 \quad (73.4)$$

In the following, we shall limit ourselves to a treatment of a plane monochromatic wave. Let us choose the x -axis in the direction of wave propagation. Equation (73.4) therefore becomes

$$\frac{\partial^2 f}{\partial x^2} + k^2 f = 0 \quad (73.5)$$

The following function will be a solution of this equation:

$$f = a \cos(\pm kx + \psi_2) \quad (73.6)$$

where a is a quantity not depending on x , and ψ_2 is a function of t .

Expressions (73.1) and (73.6) can be brought into agreement, assuming that¹

$$f = a \cos(\omega t \pm kx + \alpha) \quad (73.7)$$

where a and α are quantities depending neither on t nor on x . The different signs of kx correspond to different directions of wave propagation. We shall consider waves running in the direction of increasing x , and in this connection we shall write a minus sign in front of kx .

With an arbitrary choice of the coordinate axes, formula (73.7) becomes

$$f = a \cos (\omega t - \mathbf{k} \mathbf{r} + \alpha) \quad (73.8)$$

Any quantity characterizing a monochromatic plane wave changes according to this law, particularly the vector potential \mathbf{A} . We thus have

$$\mathbf{A} = \mathbf{A}_0 \cos (\omega t - \mathbf{k} \mathbf{r} + \alpha) \quad (73.9)$$

Here \mathbf{A}_0 is the amplitude, ω is the frequency, \mathbf{k} is the wave vector, and α is the initial phase of the wave.

The expression $(\omega t - \mathbf{k} \mathbf{r} + \alpha)$, called the **phase** of a wave, is an invariant. Indeed, the fact that the field at a given point of space at a given instant acquired, for example, a zero value $[\cos (\omega t - \mathbf{k} \mathbf{r} + \alpha) = 0]$ cannot depend on the choice of the reference frame. It thus follows that for two arbitrarily chosen reference frames, the condition

$$\omega t - \mathbf{k} \mathbf{r} + \alpha = \omega' t' - \mathbf{k}' \mathbf{r}' + \alpha'$$

must be satisfied.

If we assume that $t = 0$ and $\mathbf{r} = 0$, it can be seen from the Lorentz transformations that t' and \mathbf{r}' also equal zero. Consequently, we must have $\alpha = \alpha' = \text{inv}$, from which we conclude that the quantity

$$\Phi = \omega t - \mathbf{k} \mathbf{r} \quad (73.10)$$

also called the phase, is invariant.

For an electromagnetic wave propagating in a vacuum, the phase (73.10) can be written as

$$\Phi = \frac{\omega}{c} (ct) - \mathbf{k} \mathbf{r} = \frac{\omega}{c} x^0 - \mathbf{k} \mathbf{r} \quad (73.11)$$

Since Φ is an invariant, it follows from (73.11) that ω/c and \mathbf{k} form a four-dimensional wave vector

$$k^\mu = \left(\frac{\omega}{c}, \mathbf{k} \right) \quad (73.12)$$

This circumstance makes it possible to find the law of transformation of the wave frequency ω in passing from one inertial reference frame to another. According to the first of formulas (36.1),

$$k'^0 = \frac{k^0 - \beta k^1}{\sqrt{1 - \beta^2}}$$

whence after introducing the values $k^0 = \omega/c$, $k^1 = k_x = (\omega/c) \cos \vartheta$ (ϑ is the angle between the direction of wave propagation and the x -axis), and $\beta = v/c$ (v is the velocity of the frame

K' relative to the frame K), we find that

$$\omega' = \frac{\omega [1 - (v/c) \cos \vartheta]}{\sqrt{1 - v^2/c^2}}$$

This formula for $\vartheta = 0$ yields a formula for the longitudinal Doppler effect:

$$\omega' = \frac{\omega (1 - v/c)}{\sqrt{1 - v^2/c^2}} = \omega \sqrt{\frac{1 - v/c}{1 + v/c}}$$

When $\vartheta = \pi/2$, we arrive at a formula for the transverse Doppler effect:

$$\omega' = \frac{\omega}{\sqrt{1 - v^2/c^2}}$$

We must note that since the magnitude of the wave vector is ω/c , the square of the four-vector k^μ vanishes:

$$\sum_{\mu=0}^3 k^\mu k_\mu = (k^0)^2 - k^2 = 0$$

Let us again turn to expression (73.9). It can be written as

$$A = \text{Re} \{ \hat{A}_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \} = \text{Re} \{ \hat{A} \} \quad (73.13)$$

where \hat{A}_0 is a constant complex vector:

$$\hat{A}_0 = A_0 e^{i\alpha} \quad (73.14)$$

and \hat{A} is the complex vector in the braces [compare with (16.9) and (16.10)].

A change in the sign of the exponent in an expression such as $e^{i\varphi}$ does not change the real part of this expression. Formula (73.13) can therefore be written as follows:

$$A = \text{Re} \{ \hat{A}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \} = \text{Re} \{ \hat{A} \} \quad (73.15)$$

In this case by the phase of the wave we must understand the expression differing from (73.10) in its sign. In addition, the sign ought to be changed in the exponent of the quantity (73.14). But owing to the arbitrary nature of α , this may not be done.

Expression (73.15) is in some respects more convenient than expression (73.13) and, in addition, is more similar to the wave function describing the motion of a free particle in quantum mechanics than (73.13).

It is a simple matter to see that in performing linear operations on the quantity A determined by formula (73.15), for instance differentiation, we can perform these operations on the complex vector \hat{A} and then take the real part of the obtained quantity. For

example,

$$\frac{\partial \mathbf{A}}{\partial t} = \text{Re} \left\{ \frac{\partial \hat{\mathbf{A}}}{\partial t} \right\} = \text{Re} \{ -i\omega \hat{\mathbf{A}} \} \quad (73.16)$$

Let us calculate the curl of the vector \mathbf{A} . This is a linear operation, hence

$$\begin{aligned} [\nabla \mathbf{A}] &= \text{Re} \{ [\nabla \hat{\mathbf{A}}] \} = \text{Re} \{ [\nabla, \hat{\mathbf{A}}_0 e^{i(\mathbf{k}\mathbf{r} - \omega t)}] \} \\ &= \text{Re} \{ [\nabla e^{i(\mathbf{k}\mathbf{r} - \omega t)}, \hat{\mathbf{A}}_0] \} \end{aligned} \quad (73.17)$$

Since $\mathbf{k}\mathbf{r} = \sum k_j x_j$, we have

$$\frac{\partial}{\partial x_j} e^{i(\mathbf{k}\mathbf{r} - \omega t)} = i k_j e^{i(\mathbf{k}\mathbf{r} - \omega t)}$$

Hence,

$$\nabla e^{i(\mathbf{k}\mathbf{r} - \omega t)} = i e^{i(\mathbf{k}\mathbf{r} - \omega t)} \sum_j \mathbf{e}_j k_j = i \mathbf{k} e^{i(\mathbf{k}\mathbf{r} - \omega t)}$$

Introducing this value of the gradient into (73.17), we obtain

$$\begin{aligned} [\nabla \mathbf{A}] &= \text{Re} \{ [i \mathbf{k} e^{i(\mathbf{k}\mathbf{r} - \omega t)}, \hat{\mathbf{A}}_0] \} \\ &= \text{Re} \{ [i \mathbf{k}, \hat{\mathbf{A}}_0 e^{i(\mathbf{k}\mathbf{r} - \omega t)}] \} = \text{Re} \{ i [\mathbf{k} \hat{\mathbf{A}}] \} \end{aligned} \quad (73.18)$$

Now we can write expressions for the fields \mathbf{E} and \mathbf{B} :

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \text{Re} \left\{ i \frac{\omega}{c} \hat{\mathbf{A}} \right\} \quad (73.19)$$

$$\mathbf{B} = [\nabla \mathbf{A}] = \text{Re} \{ i [\mathbf{k} \hat{\mathbf{A}}] \} \quad (73.20)$$

[if we had proceeded from expression (73.13), a minus sign would have appeared in the last two formulas]. A comparison of them shows that the vectors \mathbf{B} and \mathbf{E} oscillate in the same phase.

Having in view that $\mathbf{k} = (\omega \sqrt{\epsilon\mu}/c) \mathbf{k}_0$, formula (73.20) can be written as

$$\mathbf{B} = \text{Re} \left\{ i \left[\sqrt{\epsilon\mu} \frac{\omega}{c} \mathbf{k}_0, \hat{\mathbf{A}} \right] \right\} = \text{Re} \left\{ \sqrt{\epsilon\mu} \left[\mathbf{k}_0, i \frac{\omega}{c} \hat{\mathbf{A}} \right] \right\}$$

The second factor in the last vector product is $\hat{\mathbf{E}}$ [see (73.19)]. Hence, for the vector $\hat{\mathbf{B}}$ whose real part gives \mathbf{B} , we obtain the relation

$$\hat{\mathbf{B}} = \sqrt{\epsilon\mu} [\mathbf{k}_0, \hat{\mathbf{E}}] \quad (73.21)$$

Introducing the notation $i(\omega/c) \hat{\mathbf{A}}_0 = \hat{\mathbf{E}}_0$, we can write expression (73.19) as follows:

$$\mathbf{E} = \text{Re} \{ \hat{\mathbf{E}}_0 e^{i(\mathbf{k}\mathbf{r} - \omega t)} \} \quad (73.22)$$

If we direct the x -axis along the vector \mathbf{k} , the vector \mathbf{E} and, consequently, $\hat{\mathbf{E}}_0$ will be in the plane yz . It can therefore be written as

a linear combination of the unit vectors \mathbf{e}_y and \mathbf{e}_z :

$$\hat{\mathbf{E}}_0 = \xi \mathbf{e}_y + \eta \mathbf{e}_z$$

where ξ and η are complex numbers.

We shall show that upon the proper choice of α , the last expression can be transformed as follows:

$$\hat{\mathbf{E}}_0 = (E_{y0}\mathbf{e}_y + E_{z0}\mathbf{e}_z) e^{-i\alpha} \quad (73.23)$$

where E_{y0} and E_{z0} are real quantities. To do this, we shall write the following expression for the square of the vector $\hat{\mathbf{E}}_0$ which, generally speaking, is a complex number:

$$\hat{\mathbf{E}}_0^2 = \text{real number} \cdot e^{-2i\alpha}$$

Let us represent the real number as the square of a vector quantity. Not only a real vector, but also the complex vector $\mathbf{a} \pm i\mathbf{b}$ may be this quantity provided that \mathbf{a} and \mathbf{b} are mutually perpendicular. Indeed, when $\mathbf{a} \cdot \mathbf{b} = 0$,

$$(\mathbf{a} + i\mathbf{b})^2 = \mathbf{a}^2 + 2i\mathbf{a} \cdot \mathbf{b} + \mathbf{b}^2 = \mathbf{a}^2 + \mathbf{b}^2 = \text{real number}$$

We have thus obtained the expression

$$\hat{\mathbf{E}}_0^2 = (\mathbf{a} + i\mathbf{b})^2 e^{-2i\alpha},$$

whence

$$\hat{\mathbf{E}}_0 = (\mathbf{a} + i\mathbf{b}) e^{-i\alpha} \quad (\mathbf{a} \perp \mathbf{b})$$

Directing the y -axis along the vector \mathbf{a} , and the z -axis along the vector \mathbf{b} , we arrive at formula (73.23).

Let us introduce expression (73.23) into formula (73.22):

$$\mathbf{E} = \text{Re} \{ (E_{y0}\mathbf{e}_y + iE_{z0}\mathbf{e}_z) e^{-i(\omega t - kx + \alpha)} \}$$

(with the directions of the axes we have chosen, $\mathbf{k}\mathbf{r} = kx$). Hence $E_y = E_{y0} \cos(\omega t - kx + \alpha)$, $E_z = E_{z0} \sin(\omega t - kx + \alpha)$ (73.24)

Inspection of these formulas shows that

$$\frac{E_y^2}{E_{y0}^2} + \frac{E_z^2}{E_{z0}^2} = 1 \quad (73.25)$$

All our reasoning and all formulas beginning with (73.22) also hold for the vector \mathbf{B} .

Our result signifies that the vector \mathbf{E} rotates in a plane perpendicular to the direction of propagation of the wave, its tip describing an ellipse (the vector \mathbf{B} behaves similarly). The direction of rotation depends on what signs (identical or opposite) E_{y0} and E_{z0} have in formulas (73.24). Such a wave is called elliptically polarized. At $E_{y0} = \pm E_{z0}$, the ellipse (73.25) transforms into a circle. In this case, the wave is circularly polarized. Finally, one of the quantities E_{y0} and E_{z0} may be zero. Here, the vector \mathbf{E} (and also \mathbf{B}) is con-

stantly directed along the same straight line. The wave in this case is called linearly polarized or plane polarized.

A monochromatic wave is thus polarized (elliptically, circularly, or linearly) in all cases. This signifies that the oscillations in the wave are ordered in some way or other.

74. A Plane Monochromatic Wave in a Conducting Medium

Equations (71.7) and (71.8) were obtained assuming that the medium contains no surplus free charges ($\rho = 0$) and no conduction currents ($\mathbf{j} = 0$). We established in the preceding section that for a plane monochromatic wave the function (73.9) with a constant amplitude is a solution of Eq. (71.7). The failure of the amplitude to depend on the coordinates signifies that the propagation of a plane wave in a dielectric is not attended by a change in its intensity.

Now let us assume that a medium has the conductivity σ so that conduction currents \mathbf{j} equal to $\sigma\mathbf{E}$ can be produced in it. We shall consider as previously that surplus free charges are absent ($\rho = 0$). With these assumptions, Maxwell's equations (55.10) and (55.11) will be written as follows:

$$[\nabla\mathbf{E}] = -\frac{1}{c} \frac{\partial\mathbf{B}}{\partial t}, \quad \nabla\mathbf{B} = 0 \quad (74.1)$$

$$[\nabla\mathbf{B}] = \frac{4\pi\mu}{c} \sigma\mathbf{E} + \frac{\varepsilon\mu}{c} \frac{\partial\mathbf{E}}{\partial t}, \quad \nabla\mathbf{E} = 0 \quad (74.2)$$

(we have substituted $\sigma\mathbf{E}$ for \mathbf{j}).

Let us take a curl of the first of Eqs. (74.1):

$$[\nabla, [\nabla\mathbf{E}]] = \frac{1}{c} \frac{\partial}{\partial t} [\nabla\mathbf{B}] \quad (74.3)$$

According to (XI.45), $[\nabla, [\nabla\mathbf{E}]] = \nabla(\nabla\mathbf{E}) - \Delta\mathbf{E}$. But $\nabla\mathbf{E} = 0$ so that only the second term equal to $-\Delta\mathbf{E}$ remains. Let us substitute it for the left-hand side of Eq. (74.3). In addition, let us introduce into the right-hand side of this equation the value of $[\nabla\mathbf{B}]$ from (74.2). The result is

$$\Delta\mathbf{E} = \frac{4\pi\mu\sigma}{c^2} \frac{\partial\mathbf{E}}{\partial t} + \frac{\varepsilon\mu}{c^2} \frac{\partial^2\mathbf{E}}{\partial t^2}$$

Let us write this equation as follows:

$$\Delta\mathbf{E} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{4\pi\mu\sigma}{c^2} \frac{\partial\mathbf{E}}{\partial t} = 0 \quad (74.4)$$

Equation (74.4) is called a **generalized wave equation**. It differs from Eq. (71.9) in an additional term containing the first time derivative of the required function. When $\sigma = 0$, this term vanishes, and Eq. (74.4) transforms into (71.9).

Considering the wave to be monochromatic, we shall seek the solution of the equation in the form

$$\mathbf{E} = \text{Re} \{ \hat{\mathbf{E}}(\mathbf{r}) e^{-i\omega t} \} = \text{Re} \{ \hat{\mathbf{E}} \} \quad (74.5)$$

where $\hat{\mathbf{E}}(\mathbf{r})$ is a complex vector function of \mathbf{r} , and $\hat{\mathbf{E}}$ is the function in braces. Differentiation of the function (74.5) with respect to t yields

$$\left. \begin{aligned} \frac{\partial \mathbf{E}}{\partial t} &= \text{Re} \{ -i\omega \hat{\mathbf{E}}(\mathbf{r}) e^{-i\omega t} \} = \text{Re} \{ -i\omega \hat{\mathbf{E}} \} \\ \frac{\partial^2 \mathbf{E}}{\partial t^2} &= \text{Re} \{ -\omega^2 \hat{\mathbf{E}}(\mathbf{r}) e^{-i\omega t} \} = \text{Re} \{ -\omega^2 \hat{\mathbf{E}} \} \end{aligned} \right\} \quad (74.6)$$

Substitution into Eq. (74.4) of the values (74.5) and (74.6), after cancelling the common factor $e^{-i\omega t}$, results in the following differential equation for $\hat{\mathbf{E}}(\mathbf{r})$:

$$\Delta \hat{\mathbf{E}}(\mathbf{r}) + \frac{\varepsilon \mu \omega^2}{c^2} \hat{\mathbf{E}}(\mathbf{r}) + i \frac{4\pi \mu \sigma \omega}{c^2} \hat{\mathbf{E}}(\mathbf{r}) = 0 \quad (74.7)$$

Let us multiply the numerator and denominator in the third term by $\varepsilon \omega$ and substitute k^2 for $\varepsilon \mu \omega^2 / c^2$ [see (73.3)]. As a result, combining the second and third terms, we get the equation

$$\Delta \hat{\mathbf{E}}(\mathbf{r}) + k^2 \left(1 + i \frac{4\pi \sigma}{\varepsilon \omega} \right) \hat{\mathbf{E}}(\mathbf{r}) = 0 \quad (74.8)$$

When $\sigma = 0$, this equation transforms into an equation like (73.5).

Let us represent the coefficient of $\hat{\mathbf{E}}(\mathbf{r})$ in Eq. (74.8) as the square of the complex wave number $\hat{k} = k_1 + ik_2$:

$$k^2 \left(1 + i \frac{4\pi \sigma}{\varepsilon \omega} \right) = (k_1 + ik_2)^2 = k_1^2 - k_2^2 + 2ik_1k_2$$

Equating the real and imaginary parts, we arrive at a system of two equations for k_1 and k_2 :

$$\left. \begin{aligned} k_1^2 - k_2^2 &= k^2 \\ 2k_1k_2 &= \frac{4\pi \sigma k^2}{\varepsilon \omega} \end{aligned} \right\} \quad (74.9)$$

The solution of this system has the form

$$\left. \begin{aligned} k_1 &= k \sqrt{\frac{1 + \sqrt{1 + (4\pi \sigma / \varepsilon \omega)^2}}{2}} \\ k_2 &= k \sqrt{\frac{-1 + \sqrt{1 + (4\pi \sigma / \varepsilon \omega)^2}}{2}} \end{aligned} \right\} \quad (74.10)$$

When $(4\pi \sigma / \varepsilon \omega)^2 \ll 1$ (i.e. when σ is small and ω is large), formulas (74.10) become greatly simplified: the quantity k_1 can be taken equal to k , and in the expression for k_2 we can assume that

$\sqrt{1 + (4\pi\sigma/\varepsilon\omega)^2} \approx 1 + \frac{1}{2} (4\pi\sigma/\varepsilon\omega)^2$, which yields the value $2\pi\sigma k/\varepsilon\omega$ for k_2 . Hence, at a low conductivity and a high frequency

$$\hat{k} = k + i \frac{2\pi\sigma k}{\varepsilon\omega} \quad (74.11)$$

Equation (74.7) can be written as

$$\Delta \hat{\mathbf{E}}(\mathbf{r}) + \hat{k}^2 \hat{\mathbf{E}}(\mathbf{r}) = 0 \quad (74.12)$$

where $\hat{k} = k_1 + ik_2$ is a complex quantity whose real and imaginary parts are determined by formulas (74.10).

Consider a plane wave. We choose the x -axis in the direction of wave propagation. Therefore $\mathbf{E}(\mathbf{r})$ depends only on x , and Eq. (74.12) can be simplified as follows:

$$\frac{\partial^2 \hat{\mathbf{E}}(x)}{\partial x^2} + \hat{k}^2 \hat{\mathbf{E}}(x) = 0$$

The following function is the solution of this equation:

$$\hat{\mathbf{E}}(x) = \hat{\mathbf{E}}_0 e^{\pm i\hat{k}x} \quad (74.13)$$

where $\hat{\mathbf{E}}_0$ is a constant complex vector. The plus and minus signs correspond to different directions of wave propagation. We shall be interested in waves running in the direction of the x -axis. We shall therefore take the sign “+” in the exponent (we wrote the factor depending on t as $e^{-i\omega t}$).

Substituting the solution (74.13) which we have found into (74.5), we obtain the following expression for \mathbf{E} :

$$\mathbf{E} = \text{Re} \{ \hat{\mathbf{E}}_0 e^{i\hat{k}x} e^{-i\omega t} \}$$

Let us represent the complex vector $\hat{\mathbf{E}}_0$ in this formula as $\mathbf{E}_0 e^{-i\alpha}$, where \mathbf{E}_0 is a real constant vector. In addition, let us express the complex number \hat{k} through its real and imaginary parts ($\hat{k} = k_1 + ik_2$). The result is

$$\mathbf{E} = \text{Re} \{ \mathbf{E}_0 e^{-i\alpha} e^{i(k_1 + ik_2)x} e^{-i\omega t} \}$$

Let us write this expression as follows:

$$\mathbf{E} = \text{Re} \{ \mathbf{E}_0 e^{-k_2 x} e^{-i(\omega t - k_1 x + \alpha)} \} \quad (74.14)$$

The real part of the expression in braces yields

$$\mathbf{E} = \mathbf{E}_0 e^{-k_2 x} \cos(\omega t - k_1 x + \alpha) \quad (74.15)$$

Expression (74.15) describes a wave whose amplitude diminishes according to the law $e^{-k_2 x}$. The phase velocity of this wave is

$$v = \frac{\omega}{k_1} \quad (74.16)$$

Hence, the real part of the complex wave number \hat{k} determines the phase velocity of the wave, and the imaginary part—the attenuation of the wave.

According to (74.10), with slight attenuation $k_1 \approx k = (\omega/c) \sqrt{\varepsilon\mu}$, and the attenuation factor is

$$k_2 \approx \frac{2\pi\sigma k}{\varepsilon\omega} = \frac{2\pi\sigma}{c} \sqrt{\frac{\mu}{\varepsilon}} \quad (74.17)$$

Let us obtain a solution of Eq. (74.7) in another way. We shall write this equation as

$$\Delta \hat{\mathbf{E}}(\mathbf{r}) + \frac{(\varepsilon + i4\pi\sigma/\omega)\mu\omega^2}{c^2} \hat{\mathbf{E}}(\mathbf{r}) = 0$$

If we introduce the complex permittivity

$$\hat{\varepsilon} = \varepsilon + i\frac{4\pi\sigma}{\omega} \quad (74.18)$$

we can write

$$\Delta \hat{\mathbf{E}}(\mathbf{r}) + \frac{\hat{\varepsilon}\mu}{c^2} \omega^2 \hat{\mathbf{E}}(\mathbf{r}) = 0$$

Let us multiply this equation by $e^{-i\omega t}$ and take into account that $\omega^2 \hat{\mathbf{E}}(\mathbf{r}) e^{-i\omega t} = -\partial^2 \hat{\mathbf{E}}/\partial t^2$, where $\hat{\mathbf{E}} = \mathbf{E}(\mathbf{r}) e^{-i\omega t}$ [see (74.6)]. As a result, we arrive at the equation

$$\Delta \hat{\mathbf{E}} - \frac{\hat{\varepsilon}\mu}{c^2} \frac{\partial^2 \hat{\mathbf{E}}}{\partial t^2} = 0 \quad (74.19)$$

differing from Eq. (71.9) only in that the electric field strength and the permittivity are not real, but complex. The field \mathbf{E} is the real part of the solution of Eq. (74.19).

We found the solution of Eq. (71.9) in Sec. 73. It is

$$\mathbf{E} = \text{Re} \{ \mathbf{E}_0 e^{-i(\omega t - kx + \alpha)} \} = \text{Re} \{ \hat{\mathbf{E}} \}$$

where $k = \sqrt{\varepsilon\mu}\omega/c$ (the wave is assumed to run in the direction of the x -axis). Consequently, when ε is real, the function

$$\hat{\mathbf{E}} = \mathbf{E}_0 e^{-i(\omega t - kx + \alpha)} \quad (74.20)$$

is the solution of Eq. (74.19).

Substitution of the complex value of the permittivity for the real one, ε , does not change the form of the solution. But the real wave number k must be replaced in it with the complex number determined by the formula

$$\hat{k} = k_1 + ik_2 = \sqrt{\hat{\varepsilon}\mu} \frac{\omega}{c} \quad (74.21)$$

where $\hat{\varepsilon}$ is the quantity from (74.18). Substitution into formula (74.20) of the complex number $\hat{k} = k_1 + ik_2$ for k leads to an expression coinciding with (74.15).

With a view to (74.18), the square of the complex number (74.21) is

$$\begin{aligned}\hat{k}^2 &= \hat{\varepsilon} \mu \omega^2 / c^2 = \left(\varepsilon + i \frac{4\pi\sigma}{\omega} \right) \mu \omega^2 / c^2 \\ &= \frac{\varepsilon \mu \omega^2}{c^2} + i \frac{4\pi\sigma \varepsilon \mu}{\varepsilon \omega} \frac{\omega^2}{c^2} = k^2 \left(1 + i \frac{4\pi\sigma}{\varepsilon \omega} \right)\end{aligned}$$

This value coincides with \hat{k}^2 in formula (74.8).

Hence, with a conducting medium, both the wave number and the permittivity become complex. The refractive index $\hat{n} = \sqrt{\hat{\varepsilon} \mu}$ is also complex. Let us write it as

$$\hat{n} = n + i\kappa = \sqrt{\hat{\varepsilon} \mu} = \sqrt{\left(\varepsilon + i \frac{4\pi\sigma}{\omega} \right) \mu} \quad (74.22)$$

Let us square this relation:

$$(n + i\kappa)^2 = n^2 - \kappa^2 + 2in\kappa = \left(\varepsilon + i \frac{4\pi\sigma}{\omega} \right) \mu = \varepsilon \mu + i \frac{4\pi\sigma \varepsilon \mu}{\omega \varepsilon}$$

We equate the real and imaginary parts:

$$n^2 - \kappa^2 = \varepsilon \mu, \quad 2n\kappa = \frac{4\pi\sigma \varepsilon \mu}{\varepsilon \omega}$$

The system of equations obtained is identical with the system (74.9). Therefore, its solutions can be found by substituting $\varepsilon \mu$ for k^2 in formulas (74.10). The result is

$$\left. \begin{aligned} n &= \sqrt{\varepsilon \mu} \sqrt{\frac{1 + \sqrt{1 + (4\pi\sigma/\varepsilon\omega)^2}}{2}} \\ \kappa &= \sqrt{\varepsilon \mu} \sqrt{\frac{-1 + \sqrt{1 + (4\pi\sigma/\varepsilon\omega)^2}}{2}} \end{aligned} \right\} \quad (74.23)$$

A comparison of formulas (74.10) and (74.23) shows that

$$k_1 = \frac{k}{\sqrt{\varepsilon \mu}} n = \frac{\omega}{c} n, \quad k_2 = \frac{k}{\sqrt{\varepsilon \mu}} \kappa = \frac{\omega}{c} \kappa \quad (74.24)$$

(we have determined k as $\omega \sqrt{\varepsilon \mu} / c$).

Substituting for k_1 its value from (74.24) into formula (74.16), we arrive at the relation

$$v = \frac{c}{n}$$

from which it follows that the real part of the complex refractive index \hat{n} is the conventional refractive index n of the medium. The

second of formulas (74.24) shows that the imaginary part of \hat{n} is proportional to the attenuation factor of the wave ($\kappa \propto k_2$).

Up to now we dealt with the electric field of a wave. The magnetic field can be obtained from relation (73.21) by introducing the complex value of the permittivity and the complex function \hat{E} into it. By (74.14), we have

$$\hat{E} = E_0 e^{-k_2 x} e^{-i(\omega t - k_1 x + \alpha)}$$

Consequently,

$$\hat{B} = \sqrt{\hat{\varepsilon}\mu} [k_0, E_0 e^{-k_2 x} e^{-i(\omega t - k_1 x + \alpha)}]$$

The complex number $\sqrt{\hat{\varepsilon}\mu}$ can be written as

$$\sqrt{\hat{\varepsilon}\mu} = n + i\kappa = \sqrt{n^2 + \kappa^2} e^{i\psi}$$

where $\psi = \tan^{-1}(\kappa/n)$. Taking advantage of this, we can write the formula

$$\hat{B} = \sqrt{n^2 + \kappa^2} [k_0, E_0 e^{-k_2 x} e^{-i(\omega t - k_1 x + \alpha - \psi)}]$$

from which it can be seen that the vectors B and E oscillate not in the same phase (as in a dielectric), but with the phase difference ψ determined by the expression

$$\tan \psi = \frac{\kappa}{n} \quad (74.25)$$

The relation between the amplitudes of the electric and magnetic fields is determined by the formula

$$B_0 = \sqrt{n^2 + \kappa^2} E_0$$

which after introduction of the values (74.23) for n and κ becomes

$$B_0 = E_0 \sqrt{\varepsilon\mu} \sqrt{1 + \left(\frac{4\pi\sigma}{\varepsilon\omega}\right)^2} \quad (74.26)$$

[compare with (73.21)].

75. Non-Monochromatic Waves ¹

Any non-monochromatic wave can be represented as the superposition of monochromatic waves of different frequencies. This operation is known as the spectral decomposition of the wave.

If the field of a wave is described by a strictly periodic function, it can be expanded into a Fourier series. In this case, the spectral expansion contains frequencies forming a discrete series of values: $\omega_1, \omega_2, \omega_3, \dots$. The frequencies ω_n ($n \neq 0$) are integral multiples

¹ Before beginning to read this section, acquaint yourself with Appendix XIV.

of the fundamental frequency ω_0 , i.e. $\omega_n = n\omega_0$. The fundamental frequency is determined by the period T of the function describing the field: $\omega_0 = 2\pi/T$.

Consider a plane non-monochromatic wave propagating in a vacuum in the positive direction of the x -axis. Any quantity characterizing such a wave (E , B , etc.) is described by the function

$$f = f\left(t - \frac{x}{c}\right) \quad (75.1)$$

[see (72.8)]. Upon fixing x , we obtain a function of t describing the oscillations of the field at a given point. We have presumed that this function is strictly periodic. Assume that its period is T :

$$f(t + T) = f(t)$$

According to (XIV.11), we can write $f(t)$ as

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{-in\omega_0 t} \quad (75.2)$$

where C_n is a constant calculated by the formula

$$C_n = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) e^{in\omega_0 t} dt$$

[see (XIV.14)].

According to formula (XIV.15), the expansion of the function $f(t)$ can also be written as

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n^* e^{in\omega_0 t} \quad (75.3)$$

The average intensity of a wave is proportional to the average value of the square of E or B , i.e. is proportional to the expression

$$\langle [f(t)]^2 \rangle = \frac{1}{T} \int_{-T/2}^{+T/2} \left(\sum_{n=-\infty}^{+\infty} C_n e^{-in\omega_0 t} \right) \left(\sum_{m=-\infty}^{+\infty} C_m^* e^{im\omega_0 t} \right) dt \quad (75.4)$$

The integrand equals $[f(t)]^2$. We have taken one of the factors in the form of (75.2), and the other in the form of (75.3), and in the second sum denoted the dummy index by the letter m instead of n .

Let us transform expression (75.4) as follows:

$$\begin{aligned} \langle [f(t)]^2 \rangle &= \sum_{n, m=-\infty}^{+\infty} C_n C_m^* \frac{1}{T} \int_{-T/2}^{+T/2} e^{i(m-n)\omega_0 t} dt \\ &= \sum_{n, m=-\infty}^{+\infty} C_n C_m^* \delta_{nm} = \sum_{n=-\infty}^{+\infty} C_n C_n^* = \sum_{n=-\infty}^{+\infty} |C_n|^2 = C_0^2 + 2 \sum_{n=1}^{\infty} |C_n|^2 \end{aligned} \quad (75.5)$$

[we have taken advantage of the orthogonal nature of the system of functions $e^{in\omega_0 t}$; see (XIV.13)].

The result we have obtained signifies that the average intensity of a non-monochromatic wave is composed of the intensities of the monochromatic components.

Now let us consider the case when a wave and, consequently, the oscillation $f(t)$ too exists during a limited time interval (and not from $t = -\infty$ to $t = +\infty$). In this case, $f(t)$ is not periodic and can be expanded not into a series, but into a Fourier integral containing a continuous series of different frequencies. By (XIV.22)

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} C_{\omega} e^{-i\omega t} d\omega \quad (75.6)$$

where C_{ω} is a function of the frequency ω determined by the expression¹

$$C_{\omega} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi) e^{i\omega \xi} d\xi \quad (75.7)$$

[see (XIV.21)]. It is evident that

$$C_{-\omega} = C_{\omega}^* \quad (75.8)$$

The total intensity of a wave during the time from $-\infty$ to $+\infty$ is determined by the expression

$$I_{\Sigma} = \int_{-\infty}^{+\infty} [f(t)]^2 dt = \int_{-\infty}^{+\infty} f(t) f(t) dt$$

Substitution of expression (75.6) for one of the factors yields

$$I_{\Sigma} = \int_{-\infty}^{+\infty} \left\{ f(t) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} C_{\omega} e^{-i\omega t} d\omega \right\} dt$$

In the formula we have obtained, it is assumed that integration is first performed over ω , and then over t . Let us change the sequence of integration, i.e. write

$$I_{\Sigma} = \int_{-\infty}^{+\infty} C_{\omega} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \right\} d\omega$$

¹ The integration variable in formula (75.7) is customarily designated by the letter t . We have used a different letter for it to stress that C_{ω} is a quantity not depending on t .

A comparison with (75.7) shows that the integral inside the braces is $C_{-\omega}$, i.e. C_{ω}^* . Consequently,

$$I_{\Sigma} = \int_{-\infty}^{+\infty} C_{\omega} C_{\omega}^* d\omega = \int_{-\infty}^{+\infty} |C_{\omega}|^2 d\omega = 2 \int_0^{\infty} |C_{\omega}|^2 d\omega \quad (75.9)$$

It follows from the formula we have obtained that the quantity $|C_{\omega}|^2$ characterizes the fraction of the total intensity per unit interval of frequencies.

The radiation of a charge experiencing deceleration can be mentioned as an example of a field that can be expanded into a Fourier integral.

Chapter XIV

RADIATION OF ELECTROMAGNETIC WAVES

76. Retarded Potentials

We assumed in the preceding chapter that charges and currents are absent, and in this connection we presumed that the right-hand side in d'Alembert's equations (71.2) and (71.3) equal zero. Now let us turn to studying time-varying fields in the presence of arbitrarily moving charges. In this case, the potentials of the field satisfy the equations

$$\Delta\varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -4\pi\rho \quad (76.1)$$

$$\Delta\mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j} \quad (76.2)$$

It is known from the theory of linear differential equations that the general solution of a non-homogeneous equation equals the sum of the general solution of a homogeneous equation and a particular solution of a non-homogeneous one. General solutions of homogeneous equations were studied in the preceding chapter. Consequently, to obtain the general solutions of Eqs. (71.1) and (71.2), it is sufficient to find their particular solutions.

Let us divide the entire space containing charges and currents into elementary volumes dV' and determine the field set up by each of the charges de contained in a given dV' . Owing to the linear nature of the equations, the required field will be the superposition of the fields set up by all the charges de .

The charge de contained in a given elementary volume dV' is, generally speaking, a function of time: $de = de(t)$. If we pay no attention to the presence of other charges de , the charge density due to the point charge being considered can be written as

$$\rho'(\mathbf{r}, t) = de(t) \delta(\mathbf{r} - \mathbf{r}') \quad (76.3)$$

where \mathbf{r}' is the position vector determining the position of the charge $de(t)$. We have designated this density by the symbol ρ' to distinguish it from the density $\rho(\mathbf{r}, t)$ determining $de(t)$ by the formula $de(t) = \rho(\mathbf{r}, t) dV'$.

Substitution of expression (76.3) into Eq. (71.3) yields

$$\Delta\varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -4\pi de(t) \delta(\mathbf{R}) \quad (76.4)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

At all points except the one for which $R = 0$, the density $\rho' = 0$, and Eq. (76.4) has the form

$$\Delta\varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \quad (76.5)$$

It is obvious that in this case the field has central symmetry relative to the point $R = 0$ and, consequently, is a function only of R . Let us therefore write Eq. (76.5) in spherical coordinates. Using expression (XI.88) for the Laplacian, we obtain

$$\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \varphi}{\partial R} \right) - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \quad (76.6)$$

We shall seek the solution in the form

$$\varphi = \frac{\psi(R, t)}{R} \quad (76.7)$$

When this condition is observed, we have

$$\begin{aligned} \frac{\partial \varphi}{\partial R} &= -\frac{\psi}{R^2} + \frac{1}{R} \frac{\partial \psi}{\partial R} \\ \frac{\partial}{\partial R} \left(R^2 \frac{\partial \varphi}{\partial R} \right) &= \frac{\partial}{\partial R} \left(-\psi + R \frac{\partial \psi}{\partial R} \right) = -\frac{\partial \psi}{\partial R} + \frac{\partial \psi}{\partial R} + R \frac{\partial^2 \psi}{\partial R^2} = R \frac{\partial^2 \psi}{\partial R^2} \\ \frac{\partial^2 \varphi}{\partial t^2} &= \frac{\partial^2}{\partial t^2} \left(\frac{\psi}{R} \right) = \frac{1}{R} \frac{\partial^2 \psi}{\partial t^2} \end{aligned}$$

The introduction of these values into Eq. (76.6) and the cancelling of $1/R$ yield the equation¹

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

As we established in Sec. 72, the general solution of such an equation has the form

$$\psi = f_1 \left(t - \frac{R}{c} \right) + f_2 \left(t + \frac{R}{c} \right) \quad (76.8)$$

[see formula (72.8); in the case we are now considering, $n = 1$].

The first term in (76.8) describes an expanding spherical wave, and the second term describes a spherical wave converging at the point $R = 0$. We are interested in a particular solution. Let us take the first term of formula (76.8) as such a solution. Introducing it into (76.7), we find an expression for φ :

$$\varphi(r, t) = \frac{f(t - R/c)}{R} \quad (76.9)$$

The solution (76.9) satisfies Eq. (76.5) with an arbitrary choice of the function $f(t - R/c)$. Let us try to choose this function so that

¹ Remember that the point for which $R = 0$ has meanwhile been omitted from consideration.

expression (76.9) will also satisfy Eq. (76.4) at the point $\mathbf{R} = 0$. We must note that at $R \rightarrow 0$ the function (76.9) tends to infinity. Consequently, its derivative with respect to the coordinates becomes very large for small R 's, so that the term $(1/c^2) (\partial^2 \varphi / \partial t^2)$ may be disregarded in comparison with $\Delta \varphi$ and Eq. (76.4) can be written as

$$\Delta \varphi = -4\pi de(t) \delta(\mathbf{R})$$

We have arrived at Poisson's equation (42.4) for the potential of a point charge. Hence, near the point $\mathbf{R} = 0$, the function (76.9) should transform into an expression of the kind $\varphi = e/R$. This will occur if we assume that $f(t - R/c) = de(t - R/c)$. Consequently, the required solution is

$$\varphi = \frac{de(t - R/c)}{R} \quad (76.10)$$

It is evident that in the direct proximity of the point $\mathbf{R} = 0$, this expression becomes

$$\varphi \approx \frac{de(t)}{R}$$

To find the potential for the arbitrary distribution of charges described by the function $\rho(\mathbf{r}, t)$, let us summate the solution (76.10) over all $de = \rho dV'$. We thus obtain

$$\varphi(\mathbf{r}, t) = \int \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) dV'}{|\mathbf{r} - \mathbf{r}'|} \quad (76.11)$$

Equation (76.2) differs from (76.1) only in the right-hand side containing \mathbf{j}/c instead of the function ρ . We can therefore directly write an expression for \mathbf{A} , by analogy with (76.11):

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int \frac{\mathbf{j}\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) dV'}{|\mathbf{r} - \mathbf{r}'|} \quad (76.12)$$

Expressions (76.11) and (76.12) are called **retarded potentials**. The name is due to the fact that the values of the potentials at the instant t are determined by the values of ρ and \mathbf{j} at earlier instants in advance of t by the retarded time $\tau = t - |\mathbf{r} - \mathbf{r}'|/c$ needed for the electromagnetic disturbance to reach the point \mathbf{r} from the point \mathbf{r}' .

To obtain the general solution of Eqs. (76.1) and (76.2), it is necessary to add to the retarded potentials the general solutions of the homogeneous equations that were found in the preceding chapter. These solutions are not related to the field produced by the system. They describe the external field acting on the system and superposed onto the field set up by the system.

In the stationary case (i.e. when ρ and \mathbf{j} do not change with time), formulas (76.11) and (76.12) transform into expressions (41.10) and (48.7), respectively.

We must note that if we had chosen the function f_2 as the particular solution [see (76.8)], we would have obtained advanced potentials instead of retarded ones. We shall not stop to consider this in greater detail.

77. Field of a Uniformly Moving Charge

Assume that the charge e is moving at the constant velocity \mathbf{v} relative to the reference frame K . Let us associate the reference frame K' with the charge. This frame, like the charge, moves relative to the frame K at the velocity \mathbf{v} .

The charge is at rest relative to the frame K' . Consequently, the potentials of the field in this frame are

$$\varphi(\mathbf{r}', t') = \frac{e}{r'}, \quad A(\mathbf{r}', t') = 0 \quad (77.1)$$

The most general solution would be $A = \text{const}$, but owing to gauge invariance, this constant could be taken equal to zero.

To find the potential in the frame K , let us transform the four-vector

$$A^\mu = (\varphi, \mathbf{A})$$

to the frame K . For this purpose, we must take the formulas (36.2) for the inverse transformation, according to which

$$\begin{aligned} \varphi(\mathbf{r}, t) &= \frac{\varphi(\mathbf{r}', t') + \beta A_x(\mathbf{r}', t')}{\sqrt{1-\beta^2}} = \frac{\varphi(\mathbf{r}', t')}{\sqrt{1-\beta^2}} \\ A_x(\mathbf{r}, t) &= \frac{\beta \varphi(\mathbf{r}', t') + A_x(\mathbf{r}', t')}{\sqrt{1-\beta^2}} = \frac{\beta \varphi(\mathbf{r}', t')}{\sqrt{1-\beta^2}} \\ A_y &= A'_y = 0, \quad A_z = A'_z = 0 \end{aligned}$$

Introducing for φ in these formulas its value from (77.1), we obtain

$$\left. \begin{aligned} \varphi(\mathbf{r}, t) &= \frac{e}{r' \sqrt{1-\beta^2}} \\ A_x(\mathbf{r}, t) &= \frac{(v/c) e}{r' \sqrt{1-\beta^2}} \end{aligned} \right\} \quad (77.2)$$

Since only the component A_x differs from zero, while the vector \mathbf{v} is directed along the x -axis, the expression for the vector potential can be written as

$$\mathbf{A}(\mathbf{r}, t) = \frac{e\mathbf{v}}{cr' \sqrt{1-\beta^2}} \quad (77.3)$$

Now in formulas (77.2) and (77.3), we must pass over from the primed coordinates (i.e. from r') to unprimed ones. According to the Lorentz transformations (35.14)

$$x' = \frac{x - vt}{\sqrt{1 - \beta^2}}, \quad y' = y, \quad z' = z$$

Consequently,

$$\begin{aligned} r' &= \sqrt{x'^2 + y'^2 + z'^2} = \sqrt{[(x - vt)/\sqrt{1 - \beta^2}]^2 + y^2 + z^2} \\ &= \frac{1}{\sqrt{1 - \beta^2}} \sqrt{(x - vt)^2 + (1 - \beta^2)(y^2 + z^2)} \end{aligned} \quad (77.4)$$

Substitution of (77.4) into (77.2) and (77.3) yields

$$\varphi(\mathbf{r}, t) = \frac{e}{\sqrt{(x - vt)^2 + (1 - \beta^2)(y^2 + z^2)}} \quad (77.5)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{e\mathbf{v}}{c \sqrt{(x - vt)^2 + (1 - \beta^2)(y^2 + z^2)}} \quad (77.6)$$

A comparison of (77.5) and (77.6) shows that the following relation exists between the potentials:

$$\mathbf{A} = \frac{\varphi\mathbf{v}}{c} \quad (77.7)$$

Formulas (77.5) and (77.6) can be simplified by expressing them in terms of the length of the vector \mathbf{R} drawn from the charge to the point of observation, and of the angle ϑ between the direction of this vector and the x -axis. If we begin to measure the time from the instant when the charge was at the origin of coordinates, we can see from Fig. 77.1 that

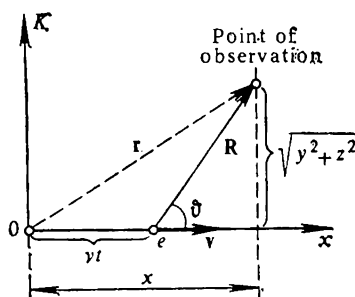


Fig. 77.1

at the origin of coordinates, we can see from Fig. 77.1 that

$$R^2 = (x - vt)^2 + y^2 + z^2$$

Therefore, by (77.4), we have

$$r' = \frac{1}{\sqrt{1 - \beta^2}} \sqrt{R^2 - \beta^2(y^2 + z^2)} = \frac{R}{\sqrt{1 - \beta^2}} \sqrt{1 - \beta^2 \frac{y^2 + z^2}{R^2}}$$

The ratio $(y^2 + z^2)/R^2$ equals $\sin^2 \vartheta$ (see Fig. 77.1). Consequently,

$$r' = \frac{R}{\sqrt{1 - \beta^2}} \sqrt{1 - \beta^2 \sin^2 \vartheta}$$

Introducing this value into (77.2) and (77.3), we obtain

$$\varphi = \frac{e}{R \sqrt{1 - \beta^2 \sin^2 \vartheta}} \quad (77.8)$$

$$\mathbf{A} = \frac{e\mathbf{v}}{cR \sqrt{1 - \beta^2 \sin^2 \vartheta}} \quad (77.9)$$

Knowing the potentials, we can use the formulas

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \quad (77.10)$$

$$\mathbf{B} = [\nabla \mathbf{A}] \quad (77.11)$$

[see (56.3) and (56.1)] to calculate the fields \mathbf{E} and \mathbf{B} . Writing (77.10) in components, we obtain

$$E_i = -\frac{\partial \varphi}{\partial x_i} - \frac{1}{c} \frac{\partial A_i}{\partial t}$$

Hence,

$$E_x = -\frac{\partial \varphi}{\partial x} - \frac{1}{c} \frac{\partial A_x}{\partial t} = -\frac{\partial \varphi}{\partial x} - \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{v\varphi}{c} \right) = -\frac{\partial \varphi}{\partial x} - \frac{v}{c^2} \frac{\partial \varphi}{\partial t} \quad (77.12)$$

$$E_y = -\frac{\partial \varphi}{\partial y}, \quad E_z = -\frac{\partial \varphi}{\partial z} \quad (77.13)$$

According to (77.5) and (77.6), φ and \mathbf{A} have the same form

$$\psi(\mathbf{r}, t) = \frac{a}{\sqrt{(x-vt)^2 + (1-\beta^2)(y^2 + z^2)}}$$

where a is a constant (scalar or vector). Introducing the notation $x - vt = \xi$, we can write that

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial \psi}{\partial \xi}, \quad \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial \xi} \frac{\partial \xi}{\partial t} = \frac{\partial \psi}{\partial \xi} (-v)$$

whence we conclude that

$$\frac{\partial \psi}{\partial t} = -v \frac{\partial \psi}{\partial x} \quad (77.14)$$

Substituting for $\partial \varphi / \partial t$ in (77.12) its value from (77.14), we arrive at the formula

$$E_x = -\frac{\partial \varphi}{\partial x} - \frac{v}{c^2} \frac{\partial \varphi}{\partial t} = -\frac{\partial \varphi}{\partial x} + \frac{v^2}{c^2} \frac{\partial \varphi}{\partial x} = -\frac{\partial \varphi}{\partial x} (1 - \beta^2) \quad (77.15)$$

Introduction of the derivatives with respect to the coordinates of expression (77.5) into formulas (77.15) and (77.13) yields

$$\left. \begin{aligned} E_x &= (1-\beta^2) \frac{e(x-vt)}{[(x-vt)^2 + (1-\beta^2)(y^2+z^2)]^{3/2}} \\ E_y &= (1-\beta^2) \frac{ey}{[(x-vt)^2 + (1-\beta^2)(y^2+z^2)]^{3/2}} \\ E_z &= (1-\beta^2) \frac{ez}{[(x-vt)^2 + (1-\beta^2)(y^2+z^2)]^{3/2}} \end{aligned} \right\} \quad (77.16)$$

These formulas can be written in the vector form:

$$\mathbf{E} = (1-\beta^2) \frac{e\mathbf{R}}{R^3 (1-\beta^2 \sin^2 \vartheta)^{3/2}} \quad (77.17)$$

where \mathbf{R} is a position vector drawn from the point where the charge is to the point of observation (see Fig. 77.1).

For the magnitude of \mathbf{E} , we have

$$E = \frac{e(1-\beta^2)}{R^2 (1-\beta^2 \sin^2 \vartheta)^{3/2}} \quad (77.18)$$

A glance at this formula shows that on the axis along which the charge is moving (i.e. at $\vartheta = 0$ and π),

$$E = \frac{e(1-\beta^2)}{R^2} \quad (77.19)$$

and in directions perpendicular to the velocity of the charge (i.e. at $\vartheta = \pi/2$),

$$E = \frac{e}{R^2 \sqrt{1-\beta^2}} \quad (77.20)$$

The field flattens, as it were, in the direction of motion of the charge, and to a greater extent with increasing v .

Let us find the magnetic field. By (77.11) and (77.7)

$$\mathbf{B} = [\nabla \mathbf{A}] = \left[\nabla, \frac{\mathbf{v}\varphi}{c} \right] = \frac{1}{c} [\nabla \varphi, \mathbf{v}] = -\frac{1}{c} [\mathbf{v}, \nabla \varphi] \quad (77.21)$$

(recall that \mathbf{v} is constant).

According to (77.10), we have

$$-\nabla \varphi = \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E} + \frac{1}{c^2} \frac{\partial (\varphi \mathbf{v})}{\partial t} = \mathbf{E} + \frac{\mathbf{v}}{c^2} \frac{\partial \varphi}{\partial t} \quad (77.22)$$

[we have taken into account relation (77.7), and also the circumstance that the velocity is independent of t].

We introduce expression (77.22) for $-\nabla \varphi$ into (77.21):

$$\mathbf{B} = \frac{1}{c} [\mathbf{v}\mathbf{E}] + \frac{1}{c} \left[\mathbf{v}, \frac{\mathbf{v}}{c^2} \frac{\partial \varphi}{\partial t} \right]$$

The second term is the vector product of collinear vectors and therefore vanishes. Hence,

$$\mathbf{B} = \frac{1}{c} [\mathbf{v}\mathbf{E}] \quad (77.23)$$

We see that the vector \mathbf{B} at each point is perpendicular both to the vector \mathbf{v} and to the vector \mathbf{E} . The vectors \mathbf{v} and \mathbf{E} are collinear on the x -axis. Consequently, on this axis, $\mathbf{B} = 0$.

If we introduce expression (77.17) for \mathbf{E} into formula (77.23), we obtain

$$\mathbf{B} = (1 - \beta^2) \frac{e}{c} \frac{[\mathbf{v}\mathbf{R}]}{R^3 (1 - \beta^2 \sin^2 \vartheta)^{3/2}} \quad (77.24)$$

When $v \ll c$, Eq. (77.24) transforms into the formula

$$\mathbf{B} = \frac{e}{c} \frac{[\mathbf{v}\mathbf{R}]}{R^3} \quad (77.25)$$

We must note that expressions (77.17) and (77.25) can be obtained with the aid of formulas (62.2) for the transformation of fields, proceeding from the fact that $\mathbf{E}' = e\mathbf{r}'/r'^3$ and $\mathbf{B}' = 0$ in the frame K' . We invite our reader to do this as an exercise.

78. Field of an Arbitrarily Moving Charge

Let us find the field of a charge moving with acceleration. Assume that motion occurs along the trajectory $\mathbf{r}_0 = \mathbf{r}_0(t)$ (Fig. 78.1).

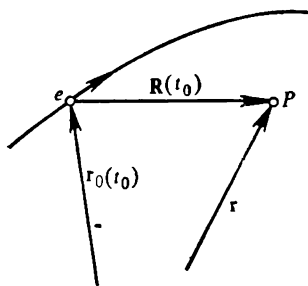


Fig. 78.1.

We shall calculate the field at the point of observation P determined by the position vector \mathbf{r} . Owing to retardation, the potentials at the point P at the instant t are determined by the position and velocity of the charge not at the same instant t , but at the earlier instant t_0 . The latter must satisfy the condition

$$c\tau = c(t - t_0) = R(t_0) \quad (78.1)$$

where $R(t_0)$ is the distance from the point at which the charge is at the instant t_0 to the point of observation P . The retardation is determined by

the time τ needed for the perturbation to reach the point \mathbf{r} from the point $\mathbf{r}_0(t_0)$.

It is a simple matter to see that $c\tau$ and the vector $\mathbf{R}(t_0)$ form the four-vector

$$R^\mu = (c\tau, \mathbf{R}) \quad (78.2)$$

Indeed, the four-vectors

$$x^\mu = \{ct, \mathbf{r}\} \quad \text{and} \quad x_0^\mu = \{ct_0, \mathbf{r}_0(t_0)\}$$

are two four-position vectors. The vector (78.2) equals the difference between these two four-vectors and, consequently, is a four-vector itself.

It can be seen from (78.4) that the square of the four-vector (78.2) is zero:

$$\sum R^\mu R_\mu = c^2 \tau^2 - R^2 = 0 \quad (78.3)$$

(this expression is in essence the square of the interval between the events of the appearance of a signal at one point and its arrival at another point).

Let us associate the reference frame K' with the charge. In this frame, the charge is at rest and, consequently, the potentials are determined by the expressions

$$\varphi'(\mathbf{r}', t') = \frac{e}{R'} = \frac{e}{c\tau'}, \quad A'(\mathbf{r}', t') = 0 \quad (78.4)$$

where \mathbf{r}' and t' are the position vector of the point and the time of observation determined in the frame K' , and R' is the distance between the point where the charge is and the point of observation. Hence, the four-potential of the field in the frame K' is

$$A'^\mu = \left(\frac{e}{c\tau'}, 0 \right) \quad (78.5)$$

Let us attempt to find an expression for the four-potential such that would transform into expression (78.5) when $\mathbf{v} = 0$. It is evident that in an arbitrary frame, the potentials should depend on the velocity \mathbf{v} of the charge. Since we are seeking a four-dimensional expression for the potential, the velocity must be taken as the four-velocity

$$u^\mu = \left(\frac{c}{\sqrt{1-v^2/c^2}}, \frac{\mathbf{v}}{\sqrt{1-v^2/c^2}} \right)$$

If we assume that the four-potential is proportional to the four-velocity, at $\mathbf{v} = 0$ the vector potential \mathbf{A} will also be zero. Further, the denominator of the time component of the four-potential (78.5) contains the time component of the four-vector (78.2). Consequently, the four-vector R^μ should be introduced into the denominator of the required expression for A^μ , and for the correct dimension to be obtained, it must be multiplied by the four-velocity (we have already introduced the four-velocity into the numerator).

■ The four-potential must thus be determined as follows:

$$A^\mu = \frac{eu^\mu}{\sum_{\nu} R^\nu u_\nu} \quad (78.6)$$

Substitution of values for R^v and u_v yields

$$\sum_v R^v u_v = \frac{1}{\sqrt{1-v^2/c^2}} (c^2\tau - R\mathbf{v})$$

Consequently, expression (78.6) can be written as

$$A^\mu = \sqrt{1-v^2/c^2} \frac{eu^\mu}{c^2\tau - R\mathbf{v}}$$

whence

$$A^\mu = \left(\frac{ec}{c^2\tau - R\mathbf{v}}, \frac{e\mathbf{v}}{c^2\tau - R\mathbf{v}} \right) \quad (78.7)$$

It is not difficult to note that when $\mathbf{v} = 0$, expression (78.7) does indeed transform into (78.5).

Taking into account that $c\tau = R$, let us write the expressions for the potentials as follows:

$$\varphi(\mathbf{r}, t) = \frac{e}{(R - R\beta)_{t_0}} \quad (78.8)$$

$$A(\mathbf{r}, t) = \frac{e\beta_{t_0}}{(R - R\beta)_{t_0}} = \varphi\beta \quad (78.9)$$

(we have used the notation $\beta = \mathbf{v}/c$).

Expressions (78.8) and (78.9) are known as the **Liénard-Wiechert potentials**. To obtain the values of the potentials at the instant t , the values of \mathbf{v} and \mathbf{R} on the right-hand sides of (78.8) and (78.9) must be taken at the instant t_0 determined by the condition

$$t_0 = t - \tau = t - \frac{R(t_0)}{c} = t - \frac{|\mathbf{r} - \mathbf{r}_0(t_0)|}{c}$$

Let us write this condition as

$$F(x, y, z, t, t_0) = t - t_0 - \frac{\{[x - x_0(t_0)]^2 + [y - y_0(t_0)]^2 + [z - z_0(t_0)]^2\}^{1/2}}{c} = 0 \quad (78.10)$$

The right-hand sides of (78.8) and (78.9) are functions of t_0 , and the latter, in turn, is a function of x, y, z , and t (here x, y, z are the coordinates of the point of observation):

$$t_0 = f(x, y, z, t) \quad (78.11)$$

Relation (78.10) is an implicit expression of the functional relation (78.11).

When finding the values of the fields by the formulas

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = [\nabla \mathbf{A}]$$

it is necessary to calculate expressions of the form $\partial\varphi/\partial x_i$, $\partial\mathbf{A}/\partial t$, and $\partial A_k/\partial x_i$, where r_i is the coordinate of the point of observation.

Since the functions φ and \mathbf{A} depend on x, y, z , in a complicated way—in terms of $t_0 = f(x, y, z, t)$, we have to perform the calculations according to the following scheme:

$$\frac{\partial \varphi}{\partial x_i} = \frac{\partial \varphi}{\partial t_0} \frac{\partial t_0}{\partial x_i} \quad (78.12)$$

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial \mathbf{A}}{\partial t_0} \frac{\partial t_0}{\partial t} \quad (78.13)$$

$$\frac{\partial \mathbf{A}_k}{\partial x_i} = \frac{\partial \mathbf{A}_k}{\partial t_0} \frac{\partial t_0}{\partial x_i} \quad (78.14)$$

It follows from relation (78.12) that

$$\nabla \varphi = \frac{\partial \varphi}{\partial t_0} \nabla t_0 \quad (78.15)$$

We shall therefore need the values of the derivatives $\partial t_0 / \partial x_i$ and $\partial t_0 / \partial t$, which can be found with the aid of relation (78.10). Let us write this relation as

$$\begin{aligned} F(x_1, x_2, x_3, t, t_0) &= t - t_0 - \frac{\{\sum [x_i - x_{0i}(t_0)]^2\}^{1/2}}{c} \\ &= t - t_0 - \frac{\{R^2\}^{1/2}}{c} = 0 \end{aligned} \quad (78.16)$$

According to the rules for the differentiation of an implicit function, we have

$$\frac{\partial t_0}{\partial x_i} = - \frac{\partial F / \partial x_i}{\partial F / \partial t_0}$$

Differentiation of (78.16) with respect to t_0 yields

$$\frac{\partial F}{\partial t_0} = -1 + \frac{\sum (x_i - x_{0i}) (\partial x_{0i} / \partial t_0)}{cR}$$

Taking into account that $x_i - x_{0i} = R_i$, and $\partial x_{0i} / \partial t_0$ is the i -th component of the velocity of the charge at the instant t_0 , we arrive at the expression

$$\frac{\partial F}{\partial t_0} = -1 + \frac{R\mathbf{v}(t_0)}{cR} = -\frac{R - R\beta}{R} \quad (78.17)$$

Similar calculations yield

$$\frac{\partial F}{\partial x_i} = -\frac{R_i}{cR} \quad (78.18)$$

Dividing (78.18) by (78.17) and reversing the sign, we obtain

$$\frac{\partial t_0}{\partial x_i} = -\frac{R_i}{c(R - R\beta)}$$

whence it follows that

$$\nabla t_0 = -\frac{R}{c(R - R\beta)} \quad (78.19)$$

Now let us find $\partial t_0 / \partial t$. According to the relevant rule for the differentiation of an implicit function, we have

$$\frac{\partial t_0}{\partial t} = -\frac{\partial F / \partial t}{\partial F / \partial t_0}$$

The derivative of the function (78.16) with respect to t is unity: $\partial F / \partial t = 1$. The derivative $\partial F / \partial t_0$ is determined by expression (78.17). Consequently,

$$\frac{\partial t_0}{\partial t} = \frac{R}{R - R\beta} \quad (78.20)$$

Now we shall commence calculating the values of $\partial \varphi / \partial t_0$ and $\partial A / \partial t_0$. By (78.8), we have

$$\frac{\partial \varphi}{\partial t_0} = -\frac{e}{(R - R\beta)^2} \left(\frac{\partial R}{\partial t_0} - \frac{\partial R}{\partial t_0} \beta - R \frac{\partial \beta}{\partial t_0} \right)$$

We can see from the relations $R = c(t - t_0)$, $R = r - r_0(t_0)$ that

$$\frac{\partial R}{\partial t_0} = -c, \quad \frac{\partial R}{\partial t_0} = -\frac{\partial r_0}{\partial t_0} = -v(t_0)$$

In addition, $\partial \beta / \partial t_0 = \dot{\beta} = \dot{v}/c$, where \dot{v} is the acceleration of the charge at the instant t_0 . Hence,

$$\frac{\partial \varphi}{\partial t_0} = -\frac{e(-c + v\beta - R\dot{\beta})}{(R - R\beta)^2} = \frac{ec(1 - \beta^2 + R\dot{\beta}/c)}{(R - R\beta)^2} \quad (78.21)$$

By (78.9), we have

$$\frac{\partial A}{\partial t_0} = \frac{\partial \varphi}{\partial t_0} \beta + \varphi \dot{\beta}$$

Substitution of expression (78.21) for $\partial \varphi / \partial t_0$ and (78.8) for φ yields

$$\frac{\partial A}{\partial t_0} = \frac{ec\beta(1 - \beta^2 + R\dot{\beta}/c)}{(R - R\beta)^2} + \frac{e\dot{\beta}}{R - R\beta} = ce \frac{\beta(1 - \beta^2 + R\dot{\beta}/c) + (\dot{\beta}/c)(R - R\beta)}{(R - R\beta)^2} \quad (78.22)$$

Finally, we can write an expression for E :

$$E = -\nabla \varphi - \frac{1}{c} \frac{\partial A}{\partial t} = -\frac{\partial \varphi}{\partial t_0} \nabla t_0 - \frac{1}{c} \frac{\partial A}{\partial t_0} \frac{\partial t_0}{\partial t}$$

[see formulas (78.15) and (78.13)]. Introducing expressions (78.21), (78.19), (78.22), and (78.20) into this equation, we obtain

$$\begin{aligned} \mathbf{E} &= -\frac{ec(1-\beta^2 + R\dot{\beta}/c)}{(R-R\beta)^2} \left[-\frac{R}{c(R-R\beta)} \right] \\ &\quad - e \frac{\beta(1-\beta^2 + R\dot{\beta}/c) + (\dot{\beta}/c)(R-R\beta)}{(R-R\beta)^2} \cdot \frac{R}{R-R\beta} \\ &= e \frac{R(1-\beta^2 + R\dot{\beta}/c) - R\beta(1-\beta^2 + R\dot{\beta}/c) - (R\dot{\beta}/c)(R-R\beta)}{(R-R\beta)^3} \quad (78.23) \end{aligned}$$

By grouping together the terms containing the acceleration $\dot{\mathbf{v}}$ of the charge (i.e. $\dot{\beta}$), we can give the expression for \mathbf{E} the form

$$\mathbf{E} = e \frac{(R-R\beta)(1-\beta^2)}{(R-R\beta)^3} + e \frac{(R\dot{\beta}/c)(R-R\beta) - (R\dot{\beta}/c)(R-R\beta)}{(R-R\beta)^3}$$

The numerator of the second term can be written as the vector triple product $[\mathbf{R}, [(R-R\beta), \dot{\beta}/c]]$. This can be verified by expanding the product according to formula (VI.5). As a result, we obtain the final expression

$$\mathbf{E} = e \frac{(R-R\beta)(1-\beta^2)}{(R-R\beta)^3} + e \frac{[\mathbf{R}, [(R-R\beta), \dot{\beta}/c]]}{(R-R\beta)^3} \quad (78.24)$$

Remember that the values of \mathbf{R} , β (i.e. \mathbf{v}), and $\dot{\beta}$ (i.e. $\dot{\mathbf{v}}$) in this formula must be taken for the instant $t_0 = t - \tau$.

The field described by expression (78.24) consists of two parts. The first depends only on the velocity of the charge and at large distances diminishes as $1/R^2$ (i.e. as a Coulomb field). The second part, besides the velocity of the charge, also depends on its acceleration and at large distances diminishes as $1/R$ (i.e. as the field strength in a spherical electromagnetic wave).

If the charge moves uniformly, the second term in formula (78.24) vanishes, and the field is determined by the expression :

$$\mathbf{E} = e \frac{(R_0 - R_0\beta)(1-\beta^2)}{(R_0 - R_0\beta)^3} \quad (78.25)$$

Here the subscript "0" on \mathbf{R} stresses the circumstance that the value of \mathbf{R} is taken for the instant t_0 . We have not used the subscript "0" on β because in uniform motion β does not depend on t .

In Sec. 77 we obtained formula (77.17) for the field of a uniformly moving charge. It can be written as

$$\mathbf{E} = e \frac{\mathbf{R}(1-\beta^2)}{R^3(1-\beta^2 \sin^2 \theta)^{3/2}} \quad (78.26)$$

Formulas (78.25) and (78.26) greatly differ from each other in their appearance. It is a simple matter to show, however, that they are

actually identical. The matter is that in formula (78.25) the field is expressed in terms of the distance from the charge to the point of observation taken at the instant $t_0 = t - \tau$. In formula (78.26), however, the field is expressed in terms of the distance from the charge to the point of observation taken at the instant of observation t .

To prove that formulas (78.25) and (78.26) are identical, let us consider Fig. 78.2. The distance OP is R_0 , eP equals R , and the segment Oe is the path travelled by the charge during the retarded time τ . Taking into account that $\tau = R_0/c$, this path can be written as $R_0\beta$.

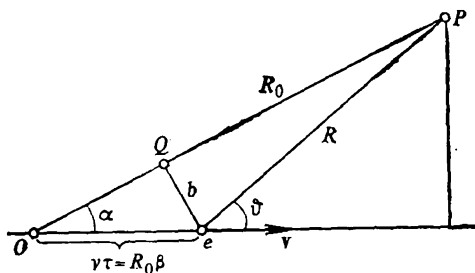


Fig. 78.2.

The vectors R_0 , R , and $R_0\beta$ are related by the expression $R = R_0 - R_0\beta$, whence the identity of the numerators of formulas (78.25) and (78.26) follows.

The length of the segment OQ equals the projection of the vector $R_0\beta$ onto the direction of the vector R_0 :

$$OQ = R_0\beta \cos \alpha = R_0\beta$$

Consequently,

$$QP = R_0 - R_0\beta \quad (78.27)$$

[compare with the denominator of formula (78.25)].

Let us express the length of the segment QP in terms of R . A glance at Fig. 78.2 shows that

$$(QP)^2 = R^2 - b^2 = R^2 - (R_0\beta \sin \alpha)^2$$

Further, it also follows from the figure that $R_0 \sin \alpha = R \sin \theta$, so that

$$(QP)^2 = R^2 - (R\beta \sin \theta)^2$$

whence

$$QP = R \sqrt{1 - \beta^2 \sin^2 \theta} \quad (78.28)$$

Finally, equating the right-hand sides of expressions (78.27) and (78.28), we arrive at the relation

$$R_0 - R_0\beta = R \sqrt{1 - \beta^2 \sin^2 \theta}$$

from which the identity of the denominators of formulas (78.25) and (78.26) follows. We have thus proved the identity of expressions (78.25) and (78.26).

Let us pass over to finding the field \mathbf{B} . By formula (XI.56), we have

$$\mathbf{B} = [\nabla \mathbf{A}] = \left[\nabla t_0, \frac{\partial \mathbf{A}}{\partial t_0} \right]$$

[\mathbf{A} depends on the coordinates in terms of t_0 , like \mathbf{a} in formula (XI.56) depends on the coordinates in terms of ξ].

Introducing (78.19) and (78.22) into the last expression, we obtain

$$\begin{aligned} \mathbf{B} &= \left[-\frac{\mathbf{R}}{c(R - R\beta)}, \quad ce \frac{\beta(1 - \beta^2 + R\dot{\beta}/c) + (\dot{\beta}/c)(R - R\beta)}{(R - R\beta)^2} \right] \\ &= \left[\frac{\mathbf{R}}{R}, \quad e \frac{-R\beta(1 - \beta^2 + R\dot{\beta}/c) - (R\dot{\beta}/c)(R - R\beta)}{(R - R\beta)^2} \right] \end{aligned}$$

Let us add in the numerator of the second factor the term $\mathbf{R}(1 - \beta^2 + R\dot{\beta}/c)$. This will not change the expression because $[\mathbf{R}\mathbf{R}] = 0$. But the second factor will now transform into \mathbf{E} [see (78.23)]. We thus arrive at the relation

$$\mathbf{B} = \left[\frac{R}{R}, \mathbf{E} \right] \quad (78.29)$$

Here $\mathbf{R} = \mathbf{R}(t_0)$. Examination of (78.29) shows that the vector \mathbf{B} at each point is perpendicular to the vector \mathbf{E} and to the vector drawn from the point where the charge was at the instant t_0 to the point of observation.

79. Field Produced by a System of Charges at Great Distances

Assume that we have a system of moving charges that do not leave the confines of a certain volume in their motion. We shall presume that the system as a whole is neutral. Let us consider the field produced by such a system at distances that are great in comparison with its dimensions. We shall place the origin of coordinates inside the system and characterize the distribution of the charge with the aid of the function $\rho = \rho(\mathbf{r}', t)$. Hence, the charge inside the volume dV' at the point with the position vector \mathbf{r}' will be $de(t) = \rho(\mathbf{r}', t) dV'$. Let \mathbf{r} stand for the position vector of the observation point P . In addition, we shall introduce the notation $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. It is obvious that \mathbf{R} is the vector drawn from de to the point P .

Let us write expressions (76.11) and (76.12) for the retarded potentials of the field produced by the system:

$$\varphi(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t - R/c) dV'}{R} \quad (79.1)$$

$$A(\mathbf{r}, t) = \frac{1}{c} \int \frac{\mathbf{j}(\mathbf{r}', t - R/c) dV'}{R} \quad (79.2)$$

According to our assumption, $r \gg r'$. Therefore, the quantity $R = |\mathbf{r} - \mathbf{r}'|$ can be considered as the value of the function $f(\mathbf{r}) = |\mathbf{r}| = r$ at the point $\mathbf{r} + \delta\mathbf{r}$, where $\delta\mathbf{r} = -\mathbf{r}'$. Taking advantage of the formula $f(\mathbf{r} + \delta\mathbf{r}) = f(\mathbf{r}) + \nabla f(\mathbf{r}) \delta\mathbf{r}$, we can write

$$R = |\mathbf{r} - \mathbf{r}'| = r + \nabla r(-\mathbf{r}') = r - \frac{\mathbf{r}}{r} \mathbf{r}' = r - \mathbf{n} \mathbf{r}' \quad (79.3)$$

where \mathbf{n} is the unit vector of the position vector \mathbf{r} .

Substitution of (79.3) for R in the formulas for the potentials yields

$$\varphi(\mathbf{r}, t) = \int \frac{\rho\left(\mathbf{r}', t - \frac{r}{c} + \frac{\mathbf{n} \mathbf{r}'}{c}\right) dV'}{r - \mathbf{n} \mathbf{r}'} \quad (79.4)$$

$$A(\mathbf{r}, t) = \frac{1}{c} \int \frac{\mathbf{j}\left(\mathbf{r}', t - \frac{r}{c} + \frac{\mathbf{n} \mathbf{r}'}{c}\right) dV'}{r - \mathbf{n} \mathbf{r}'} \quad (79.5)$$

We see that the retarded time τ consists of two parts. One of them, equal to $\tau_0 = r/c$, does not depend on \mathbf{r}' and is called the retarded time of the system. It determines the time needed for an electromagnetic perturbation to travel the path from the origin of coordinates to the point of observation. The second part, equal to $\tau = -\mathbf{n} \mathbf{r}'/c$, is called the proper retardation. It characterizes the time needed for propagation of the perturbation within the limits of the system.

Let us expand the integrand in (79.4) into a series in the ratio \mathbf{r}'/r . Considering the quantity $-\mathbf{n} \mathbf{r}'$ as the small increment δr of the argument r , we obtain

$$\begin{aligned} \frac{\rho\left(\mathbf{r}', t - \frac{r - \mathbf{n} \mathbf{r}'}{c}\right)}{r - \mathbf{n} \mathbf{r}'} &= \frac{\rho\left(\mathbf{r}', t - \frac{r}{c}\right)}{r} \\ &+ \frac{\partial}{\partial r} \left[\frac{\rho\left(\mathbf{r}', t - \frac{r}{c}\right)}{r} \right] (-\mathbf{n} \mathbf{r}') + \dots \end{aligned} \quad (79.6)$$

To see whether we can limit ourselves to the written terms in the expansion, we must assess the following terms. The latter will contain higher derivatives of ρ with respect to r . It is easy to see that the derivatives of ρ with respect to r are proportional to the derivatives of ρ with respect to t . Indeed, assuming that $t - r/c = \xi$, we

can write

$$\frac{\partial \rho}{\partial r} = \frac{\partial \rho}{\partial \xi} \frac{\partial \xi}{\partial r} = \frac{\partial \rho}{\partial \xi} \left(-\frac{1}{c} \right), \quad \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial \xi} \frac{\partial \xi}{\partial t} = \frac{\partial \rho}{\partial \xi}$$

whence

$$\frac{\partial \rho}{\partial r} = -\frac{1}{c} \frac{\partial \rho}{\partial t} \quad (79.7)$$

Similarly

$$\frac{\partial^2 \rho}{\partial r^2} = \left(-\frac{1}{c} \right)^2 \frac{\partial^2 \rho}{\partial t^2}, \quad \dots, \quad \frac{\partial^m \rho}{\partial r^m} = \left(-\frac{1}{c} \right)^m \frac{\partial^m \rho}{\partial t^m}$$

The consecutive differentiation of the function ρ/r with respect to r with the following replacement of the derivatives with respect to r with those with respect to t yields

$$\begin{aligned} \frac{\partial}{\partial r} \left(\frac{\rho}{r} \right) &= -\frac{\rho}{r^2} + \frac{1}{r} \left(-\frac{1}{c} \right) \frac{\partial \rho}{\partial t} \\ \frac{\partial^2}{\partial r^2} \left(\frac{\rho}{r} \right) &= \frac{2\rho}{r^3} - \frac{2}{r^2} \left(-\frac{1}{c} \right) \frac{\partial \rho}{\partial t} + \frac{1}{r} \left(-\frac{1}{c} \right)^2 \frac{\partial^2 \rho}{\partial t^2} \\ &\dots \dots \dots \\ \frac{\partial^m}{\partial r^m} \left(\frac{\rho}{r} \right) &= \dots \dots \dots + \frac{1}{r} \left(-\frac{1}{c} \right)^m \frac{\partial^m \rho}{\partial t^m} \end{aligned}$$

(in the last line we have written only the last term).

At large values of r , the first terms in the derivatives we have written are much smaller than the last ones. Therefore, our task consists in assessing the relative magnitude of expressions of the form

$$\frac{1}{r} \left(-\frac{1}{c} \right)^m \frac{\partial^m \rho}{\partial t^m} (-nr')^m = \frac{1}{r} \left(\frac{1}{c} \right)^m \frac{\partial^m \rho}{\partial t^m} (nr')^m \quad (79.8)$$

[in expansion (79.6), the m -th derivative is multiplied by $(\delta r)^m$].

Assume that ρ varies with time according to the harmonic law $\rho \propto \cos \omega t$. Therefore, the m -th derivative of ρ with respect to t will be of the order of $\omega^m \rho$. Substitution into (79.8) yields

$$\frac{\rho}{r} \left(\frac{\omega}{c} \right)^m (nr')^m \sim \frac{\rho}{r} \left(\frac{\omega l}{c} \right)^m$$

where l are the linear dimensions of the system of charges being considered.

The following term of the expansion will be of the order

$$\frac{\rho}{r} \left(\frac{\omega l}{c} \right)^{m+1}$$

Thus, the ratio of the consecutive terms of expansion (79.6) in the order of its magnitude is $\omega l/c$. Replacing the frequency with the period of the variation of ρ (by the formula $\omega = 2\pi/T$), we obtain

$$\frac{\omega l}{c} = \frac{2\pi l}{cT} \sim \frac{l}{cT}$$

What has been said above shows that the subsequent terms in the expansion (79.6) may be disregarded if the condition

$$\frac{l}{cT} \ll 1 \quad (79.9)$$

is satisfied. The ratio l/c determines the proper retardation time τ' . Consequently, condition (79.9) can be written as

$$\tau' \ll T \quad (79.10)$$

It can be seen from (79.10) that we may limit ourselves to the first terms of expansion (79.6) when the time needed for the propagation of an electromagnetic perturbation within the limits of the system is much smaller than the time during which the distribution of the charges in the system changes appreciably.

Condition (79.9) can be written in two other ways. The product cT gives the wavelength λ of the radiation produced by the system. Therefore, inequality (79.9) can be written as

$$l \ll \lambda \quad \text{is much less than} \quad (79.11)$$

the dimensions of the system must be much smaller than the wavelength.

Finally, having in view that l/T in the order of magnitude equals the velocity v of the charges in the system, instead of inequality (79.9) we can write

$$v \ll c \quad (79.12)$$

A glance at the last relation shows that by interrupting expansion (79.6) at the second term, we have limited ourselves to considering the radiation of a non-relativistic system of charges.

Let us again turn to the calculation of the potentials, assuming conditions (79.10)-(79.12) to be satisfied. The substitution of (79.6) into (79.4) yields

$$\varphi(\mathbf{r}, t) = \frac{1}{r} \int \rho\left(\mathbf{r}', t - \frac{r}{c}\right) dV' - \frac{\partial}{\partial r} \left\{ \frac{1}{r} \int \rho\left(\mathbf{r}', t - \frac{r}{c}\right) \mathbf{nr}' dV' \right\} \quad (79.13)$$

(we remind our reader that integration is performed over the primed coordinates, therefore r can be put outside the integral; in addition, we have changed the sequence of differentiation with respect to r and of integration).

The density of the charge at the instant $t - (r/c)$ is inside the first integral. Consequently, this integral gives the total charge of the system, which owing to the presumed electroneutrality of the system is zero. We must therefore retain only the second term in formula (79.13). Putting \mathbf{n} in it outside the integral and the deriva-

tive, we arrive at the expression

$$\varphi(\mathbf{r}, t) = -n \frac{\partial}{\partial r} \left\{ \frac{1}{r} \int \rho(\mathbf{r}', t - \frac{r}{c}) \mathbf{r}' dV' \right\}$$

The integral in this expression is the dipole electric moment which the system had at the instant $t - r/c$: dipole moment

$$\mathbf{p}(t - \frac{r}{c}) = \int \rho(\mathbf{r}', t - \frac{r}{c}) \mathbf{r}' dV' \quad \vec{p} = \int \rho(\vec{r}) \vec{r} dV$$

[compare with (43.6)]. We can therefore write

$$\varphi(\mathbf{r}, t) = -n \frac{\partial}{\partial r} \left[\frac{\mathbf{p}(t - r/c)}{r} \right] \quad (79.14)$$

Finally, having performed differentiation and taken into consideration that $\partial \mathbf{p} / \partial r = -(1/c) \partial \mathbf{p} / \partial t = -(1/c) \dot{\mathbf{p}}$ [compare with (79.7)], we obtain

$$\varphi(\mathbf{r}, t) = \frac{n \mathbf{p}(t - r/c)}{r^2} + \frac{n \dot{\mathbf{p}}(t - r/c)}{cr}$$

The first term in this formula coincides with the potential (43.9) of a static dipole ($\mathbf{n} = \mathbf{r}/r$). We must note that the field corresponding to this term at the distance r and the instant t is determined by the value of the dipole moment at the instant $t - r/c$. The first term diminishes with an increasing distance r much more rapidly than the second term. Therefore, considering the field at great distances, we can assume that

$$\varphi(\mathbf{r}, t) = \frac{n \dot{\mathbf{p}}(t - r/c)}{cr} \quad (79.15)$$

Let us go over to determination of the vector potential. Formula (79.2) differs from (79.1) only in that $\mathbf{j}(\mathbf{r}', t - R/c)$ is inside the integral instead of $\rho(\mathbf{r}', t - R/c)$. Consequently, by expanding the integrand into a series, we obtain an expression similar to (79.13):

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{cr} \int \mathbf{j}(\mathbf{r}', t - \frac{r}{c}) dV' - \frac{\partial}{\partial r} \left\{ \frac{1}{cr} \int \mathbf{j}(\mathbf{r}', t - \frac{r}{c}) (nr') dV' \right\} \quad (79.16)$$

If the currents were stationary, i.e. did not depend on t , the first integral would vanish [see (51.5)]. For non-stationary currents, however, this integral differs from zero. We may therefore retain only the first term in expansion (79.16)¹. We can thus assume that

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{cr} \int \mathbf{j}(\mathbf{r}', t - \frac{r}{c}) dV' \quad (B)$$

¹ In formula (79.13), we could not disregard the second term because the first one vanished.

We shall prove that $\int \mathbf{j}(\mathbf{r}', t - r/c) dV'$ equals the time derivative of the dipole moment of the system taken for the instant $t - r/c$. It will be the simplest to prove this by passing from the continuous distribution of the charges to a discrete one. Let us perform the substitution

$$\int \mathbf{j} dV' = \int \rho \mathbf{v} dV' \rightarrow \sum e_a \mathbf{v}_a$$

(the velocities of the charges, like the function $\rho \mathbf{v}$, must be taken for the instant $t - r/c$). However,

$$\sum e_a \mathbf{v}_a = \sum e_a \dot{\mathbf{r}}'_a = \frac{d}{dt} \sum e_a \mathbf{r}'_a = \dot{\mathbf{p}}(t - r/c)$$

Consequently,

$$\mathbf{A}(\mathbf{r}, t) = \frac{\dot{\mathbf{p}}(t - r/c)}{cr} \quad (79.17)$$

A comparison with (79.15) allows us to write

$$\varphi = \mathbf{A} \mathbf{n} \quad (79.18)$$

The potentials (79.15) and (79.17) are determined by the value of the time derivative of the dipole moment of the system. This is why they are called potentials calculated in a dipole approximation. The dipole approximation is allowable when the conditions (79.10)-(79.12) are observed.

80. Dipole Radiation

The region of a field that is at a distance r from the radiating system much greater not only than the dimensions of the system l , but also than the radiated wavelength ($r \gg \lambda \gg l$) is known as a **wave zone**.

In this zone, conditions are observed in which the dipole approximation treated in Sec. 79 holds. In this approximation

$$\varphi = \frac{\dot{\mathbf{n}} \mathbf{p}(t - r/c)}{cr}, \quad \mathbf{A} = \frac{\dot{\mathbf{p}}(t - r/c)}{cr} \quad (80.1)$$

[see (79.15) and (79.17)].

To calculate \mathbf{E} , we must find $\nabla \varphi$ and $\partial \mathbf{A} / \partial t$. Using formula (XI.51), we obtain

$$\nabla \varphi = \frac{\partial \varphi}{\partial r} \nabla r = \frac{\partial \varphi}{\partial r} \mathbf{n}$$

(recall that $\mathbf{n} = \mathbf{e}_r = \mathbf{r}/r$). Hence,

$$\nabla \varphi = \frac{\partial}{\partial r} \left(\frac{\dot{\mathbf{n}} \mathbf{p}(t - r/c)}{cr} \right) \mathbf{n} = -\frac{\dot{\mathbf{n}} \mathbf{p}}{cr^2} \mathbf{n} - \frac{\ddot{\mathbf{n}} \mathbf{p}}{c^2 r} \mathbf{n} \quad (80.2)$$

[we have taken advantage of the circumstance that $\partial \dot{\mathbf{p}}/\partial r = -(1/c) \partial \dot{\mathbf{p}}/\partial t$].

The first term in the expression we have obtained diminishes with an increasing distance much more rapidly than the second one. It may therefore be ignored for great distances, and we may consider that

$$\nabla\varphi = -\frac{\ddot{\mathbf{n}}\dot{\mathbf{p}}(t-r/c)}{c^2r} \mathbf{n}$$

The derivative $\partial \mathbf{A}/\partial t = \ddot{\mathbf{p}}/cr$. Hence,

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \frac{(\ddot{\mathbf{n}}\dot{\mathbf{p}}) \mathbf{n}}{c^2r} - \frac{1}{c} \frac{\ddot{\mathbf{p}}}{cr} = \frac{(\ddot{\mathbf{n}}\dot{\mathbf{p}}) \mathbf{n} - \ddot{\mathbf{p}}}{c^2r}$$

The numerator of this expression can be written as $[\mathbf{n} [\ddot{\mathbf{n}}\dot{\mathbf{p}}]]$. This can readily be verified by expanding the vector triple product using formula (VI.5) and taking into account that $\mathbf{nn} = 1$. The electric field is thus determined by the formula

$$\mathbf{E} = \frac{1}{c^2r} [\mathbf{n}, [\ddot{\mathbf{n}}\dot{\mathbf{p}}]] = \frac{1}{c^2r} [[\ddot{\mathbf{p}}\mathbf{n}], \mathbf{n}]$$

Let us go over to calculation of the magnetic field. The vector potential is a function of r . Therefore, by (XI.56), we have

$$\mathbf{B} = [\nabla \mathbf{A}] = \left[\nabla r, \frac{\partial \mathbf{A}}{\partial r} \right] = \left[\mathbf{n}, \frac{\partial \mathbf{A}}{\partial r} \right] \quad (80.3)$$

Differentiation of expression (80.1) for \mathbf{A} yields

$$\frac{\partial \mathbf{A}}{\partial r} = \frac{\partial}{\partial r} \left(\frac{\dot{\mathbf{p}}(t-r/c)}{cr} \right) = -\frac{\dot{\mathbf{p}}}{cr^2} - \frac{\ddot{\mathbf{p}}}{c^2r}$$

[compare with (80.2)]. Discarding the term proportional to $1/r^2$, we find that $\partial \mathbf{A}/\partial r = -\ddot{\mathbf{p}}/c^2r$. Hence,

$$\mathbf{B} = -\frac{1}{c^2r} [\ddot{\mathbf{n}}\dot{\mathbf{p}}] = \frac{1}{c^2r} [\ddot{\mathbf{p}}\mathbf{n}]$$

We shall write the final expressions for \mathbf{E} and \mathbf{B} :

$$\mathbf{E} = \frac{1}{c^2r} [[\ddot{\mathbf{p}}\mathbf{n}], \mathbf{n}], \quad \mathbf{B} = \frac{1}{c^2r} [\ddot{\mathbf{p}}\mathbf{n}] \quad (80.4)$$

(remember that the values of $\ddot{\mathbf{p}}$ must be taken for the instant $t - r/c$). A comparison of these expressions leads to the conclusion that

$$\mathbf{E} = [\mathbf{B}\mathbf{n}] \quad (80.5)$$

whence it follows that the vector \mathbf{E} is perpendicular to the vector \mathbf{B} . Examination of expressions (80.4) shows that the vectors \mathbf{E} and \mathbf{B} are perpendicular to the vector \mathbf{n} [the perpendicularity of \mathbf{E} to \mathbf{n}

can also be seen from (80.5)]. Hence, as in a plane wave, the vectors \mathbf{B} , \mathbf{E} , and \mathbf{n} are mutually perpendicular [see formula (72.16)¹]. In addition, the vectors \mathbf{B} and \mathbf{E} , as in a plane wave, are identical in magnitude, and

$$E = B = \frac{|\ddot{\mathbf{p}}| \sin \vartheta}{c^2 r} \quad (80.6)$$

where ϑ is the angle between the directions of the vectors $\ddot{\mathbf{p}}$ and \mathbf{n} .

That the relations observed for a plane wave were found to hold for the field we are studying is not surprising. At distances that are great in comparison with the dimensions of the radiating system, a wave must be spherical. At the same time, provided that $r \gg \lambda$, small portions of the spherical wave virtually coincide with a plane wave.

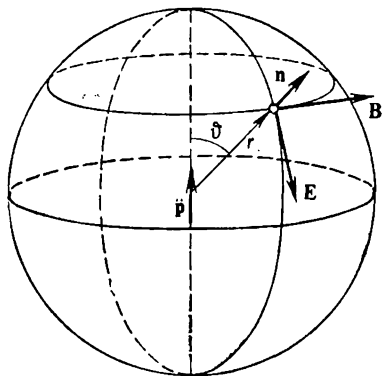


Fig. 80.1.

It can be seen from (80.4) that the fields \mathbf{E} and \mathbf{B} are determined by the second derivative of the dipole moment of the system. This is the reason why the radiation being considered is called dipole radiation.

The dipole moment is determined by the expression $\mathbf{p} = \sum e\mathbf{r}'$.

Consequently, $\ddot{\mathbf{p}} = \sum e\ddot{\mathbf{r}}' = \sum e\ddot{\mathbf{v}}'$. It thus follows that charges emit electromagnetic waves only when they move with acceleration.

To comprehend the pattern of the field at great distances, let us introduce a spherical system of coordinates, measuring the polar angle ϑ from the direction of the vector $\ddot{\mathbf{p}}(t - r/c)$ (Fig. 80.1). By (80.4), the vector \mathbf{B} is perpendicular to the plane determined by the vectors $\ddot{\mathbf{p}}$ and \mathbf{n} . Consequently, \mathbf{B} is directed along a tangent to a "parallel", the vectors $\ddot{\mathbf{p}}$, \mathbf{n} , and \mathbf{B} forming a right-handed system.

Examination of (80.5) shows that the vectors \mathbf{B} , \mathbf{n} , and \mathbf{E} form a right-handed system. Hence, it follows that \mathbf{E} is directed along a tangent to a "meridian", the directions of $\ddot{\mathbf{p}}$ and \mathbf{E} on the equator being opposite. We stress once more that the vectors depicted in Fig. 80.1 relate to different instants: $\ddot{\mathbf{p}}$ to the instant $t - r/c$, and \mathbf{B}

¹ We multiply relation (80.5) by \mathbf{n} and use formula (VI.5):

$$[\mathbf{nE}] = [\mathbf{n}, [\mathbf{Bn}]] = \mathbf{B}(\mathbf{nn}) - \mathbf{n}(\mathbf{nB}) = \mathbf{B}$$

We have arrived at formula (72.16) (in a vacuum $\sqrt{\epsilon\mu} = 1$, $\mathbf{H} = \mathbf{B}$).

and \mathbf{E} to the instant t . The magnitudes of the vectors \mathbf{B} and \mathbf{E} are proportional to $\sin \vartheta$ [see (80.4)]. Therefore, the fields have the maximum value at the "equator" and vanish at the "poles".

To determine the intensity of radiation in different directions and the total radiated power, let us calculate the Poynting vector. With a view to (80.6), we obtain

$$\mathbf{S} = \frac{c}{4\pi} [\mathbf{E}\mathbf{B}] = \frac{c}{4\pi} E B \mathbf{n} = \frac{\dot{\mathbf{p}}^2 \sin^2 \vartheta}{4\pi c^3 r^2} \mathbf{n} \quad (80.7)$$

Hence, the intensity of dipole radiation is proportional to $\sin^2 \vartheta$. The diagram showing the intensity as a function of ϑ has a two-lobed pattern.

To determine the radiated power P , let us find the energy flux through the entire spherical surface. The area of a spherical band of width $d\vartheta$ is $2\pi r^2 \sin \vartheta d\vartheta$. Consequently,

$$P = \int \mathbf{S} d\mathbf{f} = \int_0^\pi \frac{\dot{\mathbf{p}}^2 \sin^2 \vartheta}{4\pi c^3 r^2} 2\pi r^2 \sin \vartheta d\vartheta = \frac{2\dot{\mathbf{p}}^2}{3c^3} \quad (80.8)$$

Assume that of all the charges of the system only one has acceleration. Hence, $\dot{\mathbf{p}} = \sum e_a \dot{\mathbf{v}}_a = e\dot{\mathbf{v}}$ and the radiated power is

$$P = \frac{2e^2 \dot{\mathbf{v}}^2}{3c^3} \quad (80.9)$$

This formula also holds when there is only one charge moving with the acceleration $\dot{\mathbf{v}}$.

81. Magnetic Dipole and Quadrupole Radiations

If the properties of a system of charges are such that $\ddot{\mathbf{p}} = 0$, no dipole radiation is produced. This does not signify, however, that there is no radiation at all. In this case, we must take into account the terms of the expansion of the potentials which we disregarded in the dipole approximation.

We established in the preceding section that in the wave zone a wave in small regions is close to a plane wave, for which relation (80.5) holds. This relation makes it quite simple to find \mathbf{E} if we know \mathbf{B} . To find \mathbf{B} , on the other hand, it is sufficient to know only \mathbf{A} . We shall therefore limit ourselves to finding the vector potential.

Consider the second term of formula (79.16) which we disregarded in the dipole approximation. Since in the case we are interested in the first term in (79.16) is zero, the vector potential is

$$\mathbf{A}(\mathbf{r}, t) = -\frac{\partial}{\partial r} \left\{ \frac{1}{cr} \int \mathbf{j} \left(\mathbf{r}', t - \frac{r}{c} \right) (\mathbf{n}\mathbf{r}') dV' \right\}$$

Differentiation with respect to r yields two terms. One of them is proportional to $1/r^2$, and the second is proportional to $1/r$. The first term diminishes with the increasing distance much more rapidly than the second one. We therefore disregard it, as we have already done many times. In addition, we take into account that $\partial \mathbf{j} / \partial r = -(1/c) \partial \mathbf{j} / \partial t$. The result is

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c^2 r} \frac{\partial}{\partial t} \int \mathbf{j}(\mathbf{r}', t - r/c) (\mathbf{nr}') dV'$$

To simplify our further calculations, let us pass over from a continuous distribution of the charges to a discrete one. The expression for \mathbf{A} therefore becomes

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c^2 r} \frac{\partial}{\partial t} \sum_a e_a \mathbf{v}_a (\mathbf{nr}'_a) \quad (81.1)$$

The values of \mathbf{v}_a and \mathbf{r}'_a are taken for the instant $t - r/c$.

The expression $\mathbf{v}_a (\mathbf{nr}'_a)$ can be written as follows:

$$\mathbf{v}_a (\mathbf{nr}'_a) = \frac{\partial}{\partial t} \{ \mathbf{r}'_a (\mathbf{nr}'_a) \} - \mathbf{r}'_a (\mathbf{nv}_a) \quad (81.2)$$

Let us divide $\mathbf{v}_a (\mathbf{nr}'_a)$ into two equal parts and replace one of them with half of expression (81.2):

$$\mathbf{v}_a (\mathbf{nr}'_a) = \frac{1}{2} \mathbf{v}_a (\mathbf{nr}'_a) + \frac{1}{2} \frac{\partial}{\partial t} \{ \mathbf{r}'_a (\mathbf{nr}'_a) \} - \frac{1}{2} \mathbf{r}'_a (\mathbf{nv}_a)$$

The first and third terms can be written as a vector triple product: $\frac{1}{2} [\mathbf{n}, [\mathbf{v}_a, \mathbf{r}'_a]]$; this can be verified with the aid of formula (VI.5). Hence,

$$\mathbf{v}_a (\mathbf{nr}'_a) = \frac{1}{2} [\mathbf{n}, [\mathbf{v}_a \mathbf{r}'_a]] + \frac{1}{2} \frac{\partial}{\partial t} \{ \mathbf{r}'_a (\mathbf{nr}'_a) \}$$

Substitution of this expression into formula (81.1) yields

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \frac{1}{2c^2 r} \frac{\partial}{\partial t} \sum_a e_a [\mathbf{n}, [\mathbf{v}_a \mathbf{r}'_a]] \\ &\quad + \frac{1}{2c^2 r} \frac{\partial^2}{\partial t^2} \sum_a e_a \mathbf{r}'_a (\mathbf{nr}'_a) = \mathbf{A}_m + \mathbf{A}_Q \end{aligned} \quad (81.3)$$

The meaning of the subscripts m and Q will be revealed below.

Let us put \mathbf{n} outside the sum sign in the first term and exchange the places of \mathbf{v}_a and \mathbf{r}'_a . As a result, this term becomes

$$\mathbf{A}_m = -\frac{1}{2c^2 r} \frac{\partial}{\partial t} \left[\mathbf{n}, \sum_a e_a [\mathbf{r}'_a \mathbf{v}_a] \right] = \frac{1}{cr} \frac{\partial}{\partial t} \left[\frac{1}{2c} \sum_a e_a [\mathbf{r}'_a \mathbf{v}_a], \mathbf{n} \right]$$

The first factor is the magnetic moment \mathbf{m} of the system [see formula (51.15)]. We can therefore write that

$$\mathbf{A}_m = \frac{1}{cr} [\dot{\mathbf{m}}\mathbf{n}] \quad (81.4)$$

where $\dot{\mathbf{m}} = \partial\mathbf{m}/\partial t$ is taken for the instant $t - r/c$. The potential \mathbf{A}_m is thus determined by the changes in the magnetic dipole moment. Therefore, the relevant radiation is called magnetic-dipole radiation.

For magnetic-dipole radiation, the magnetic field is

$$\mathbf{B}_m = [\nabla\mathbf{A}_m] = \left[\nabla r, \frac{\partial\mathbf{A}_m}{\partial r} \right] = \left[\mathbf{n}, \frac{\partial\mathbf{A}_m}{\partial r} \right]$$

[compare with (80.3)]. If we disregard in $\partial\mathbf{A}_m/\partial r$ the term proportional to $1/r^2$ and replace $\partial/\partial r$ with $(-1/c) \partial/\partial t$ [recall that $\ddot{\mathbf{m}} = \ddot{\mathbf{m}}(t - r/c)$], we obtain

$$\mathbf{B}_m = -\frac{1}{c^2 r} [\mathbf{n}, [\ddot{\mathbf{m}}\mathbf{n}]] = \frac{1}{c^2 r} [[\ddot{\mathbf{m}}\mathbf{n}], \mathbf{n}]$$

Using relation (72.16) equivalent to (80.5), we find that

$$\mathbf{E}_m = -\frac{1}{c^2 r} [\ddot{\mathbf{m}}\mathbf{n}] = \frac{1}{c^2 r} [\mathbf{n}\ddot{\mathbf{m}}]$$

Hence, the fields in magnetic-dipole radiation are

$$\left. \begin{aligned} \mathbf{E}_m &= \frac{1}{c^2 r} [\mathbf{n}\ddot{\mathbf{m}}], \\ \mathbf{B}_m &= \frac{1}{c^2 r} [[\ddot{\mathbf{m}}\mathbf{n}], \mathbf{n}] \end{aligned} \right\} \quad (81.5)$$

A comparison of the results obtained with formulas (80.4) for dipole radiation shows that \mathbf{B}_m is expressed in terms of $\ddot{\mathbf{m}}$ by a formula similar to the one expressing \mathbf{E} in terms of $\ddot{\mathbf{p}}$. The formulas for \mathbf{E}_m and \mathbf{B} differ, apart from the substitution of $\ddot{\mathbf{m}}$ for $\ddot{\mathbf{p}}$, in the sign. It thus follows that a pattern of the field in magnetic-dipole radiation in the wave zone can be obtained by substituting $\ddot{\mathbf{m}}$ for $\ddot{\mathbf{p}}$, \mathbf{B}_m for \mathbf{E} , and $-\mathbf{E}_m$ for \mathbf{B} in Fig. 80.1.

Let us turn to the second term of formula (81.3), i.e. to the expression

$$\mathbf{A}_Q = \frac{1}{2c^2 r} \frac{\partial^2}{\partial t^2} \sum_a e_a \mathbf{r}'_a (\mathbf{n} \mathbf{r}'_a) \quad (81.6)$$

The addition to \mathbf{A}_Q of the expression $f(r) \mathbf{n}$ [here $f(r)$ is any function of r] will not change \mathbf{B}_Q because $[\nabla, f(r) \mathbf{n}] = 0$ [see (XI.54)].

Let us take as $f(r)$ the function

$$f(r) = -\frac{1}{6c^2r} \frac{\partial}{\partial t^2} \sum_a e_a r_a'^2$$

[r_a' is a function of the argument $(t - r/c)$, which when acted upon by the operator ∇ behaves like a function of r]. Multiplication of this function by \mathbf{n} and the addition of it to (81.6) yields

$$\mathbf{A}_Q = \frac{1}{6c^2r} \frac{\partial^2}{\partial t^2} \sum_a e_a \{3\mathbf{r}_a'(\mathbf{n}\mathbf{r}_a') - \mathbf{r}_a'^2\mathbf{n}\}$$

Let us write an expression for the i -th component of the vector \mathbf{A}_Q :

$$A_{Qi} = \frac{1}{6c^2r} \frac{\partial^2}{\partial t^2} \sum_a e_a \left\{ 3x_{ai}' \left(\sum_k n_k x_{ak}' \right) - r_a'^2 n_i \right\}$$

The last term in the braces can be written as $\sum_k r_a'^2 \delta_{ik} n_k$. Hence, A_{Qi} will appear as follows:

$$\begin{aligned} A_{Qi} &= \frac{1}{6c^2r} \frac{\partial^2}{\partial t^2} \sum_a e_a \left\{ 3x_{ai}' \left(\sum_k n_k x_{ak}' \right) - \left(\sum_k r_a'^2 \delta_{ik} n_k \right) \right\} \\ &= \frac{1}{6c^2r} \frac{\partial^2}{\partial t^2} \sum_k n_k \sum_a e_a \{ 3x_{ai}' x_{ak}' - r_a'^2 \delta_{ik} \} \end{aligned}$$

But the sum over a is Q_{ik} — a component of the tensor of the quadrupole moment of the system [see formula (43.13)]. Consequently

$$A_{Qi} = \frac{1}{6c^2r} \frac{\partial^2}{\partial t^2} \sum_k Q_{ik} n_k = \frac{1}{6c^2r} \frac{\partial^2}{\partial t^2} Q_i$$

where Q_i is a component of the vector obtained when the vector \mathbf{n} is multiplied by the tensor Q_{ik} . Denoting this vector by the symbol \mathbf{Q} , we can write

$$\mathbf{A}_Q = \frac{1}{6c^2r} \ddot{\mathbf{Q}} \quad (81.7)$$

The potential \mathbf{A}_Q is determined by the changes in the quadrupole moment of the system. For this reason, the corresponding radiation is called quadrupole radiation.

The magnetic field of quadrupole radiation is

$$\mathbf{B}_Q = [\nabla \mathbf{A}_Q] = \left[\nabla r, \frac{\partial \mathbf{A}_Q}{\partial r} \right] = \left[\mathbf{n}, \frac{\partial \mathbf{A}_Q}{\partial r} \right]$$

In calculating $\partial \mathbf{A}_Q / \partial r$, we disregard the term proportional to $1/r^2$ and replace the derivative with respect to r with one with respect to t . The result is

$$\mathbf{B}_Q = \frac{1}{6c^3r} [\ddot{\mathbf{Q}} \mathbf{n}] \quad (81.8)$$

(we have transposed the factors to avoid writing the minus sign introduced by the factor $-1/c$).

Taking advantage of relation (80.5), we find that

$$E_Q = \frac{1}{6c^3 r} [[\ddot{\mathbf{Q}} \mathbf{n}], \mathbf{n}] \quad (81.9)$$

In comparison with formulas (80.4), expressions (81.8) and (81.9) have the additional factor $1/6c$. In addition, they contain $\ddot{\mathbf{Q}}$ instead of $\ddot{\mathbf{p}}$. Otherwise, the formulas for the fields in quadrupole radiation are identical to the relevant formulas for dipole radiation.

Calculations show that the total power of all three kinds of radiation (including dipole radiation) is determined by the expression

$$P = \frac{2}{3c^3} \ddot{\mathbf{p}}^2 + \frac{2}{3c^3} \ddot{\mathbf{m}}^2 + \frac{1}{180c^5} \sum_{i, k} \ddot{\mathbf{Q}}_{ik}^2 \quad (81.10)$$

Let us assess the relative intensity of various kinds of radiation. We shall assume for simplicity that the charges of the system move according to a harmonic law. Hence

$$r' = l \cos \omega t, \quad v = l\omega \sin \omega t$$

where l is a quantity of the order of the system's dimensions. The average values of the magnitudes of $\cos \omega t$ and $\sin \omega t$ are quantities of the order of unity. Therefore in the final expressions characterizing the order of magnitude of the expressions being considered, we shall discard the factors $\cos \omega t$ and $\sin \omega t$. For instance, if we do not have in mind the necessity of time differentiation, we can write that

$$r' \sim l, \quad v \sim l\omega \quad (81.11)$$

It can be seen from definition (43.5) that p is a quantity of the order of er' , i.e.

$$p \sim el \cos \omega t, \quad \dot{p} \sim el\omega \sin \omega t, \quad \ddot{p} \sim el\omega^2 \cos \omega t$$

Hence we find that with respect to the order of magnitude, the power of dipole radiation is determined by the expression

$$P_p \sim \frac{\ddot{p}^2}{c^3} \sim \frac{e^2 l^2 \omega^4}{c^3} \sim \frac{e^2 v^2 \omega^2}{c^3} \quad (81.12)$$

[see (81.11)].

It follows from definition (51.15) that the magnetic moment has a magnitude of the order of $er'v/c$, i.e.

$$m \sim \frac{1}{c} el^2 \omega \cos \omega t \sin \omega t \sim \frac{1}{c} el^2 \omega \sin 2\omega t$$

$$\dot{m} \sim \frac{1}{c} el^2 \omega^2 \cos 2\omega t, \quad \ddot{m} \sim \frac{1}{c} el^2 \omega^3 \sin 2\omega t$$

whence

$$P_m \sim \frac{\ddot{m}^2}{c^3} \sim \frac{e^2 l^4 \omega^6}{c^5} \sim \frac{e^2 v^4 \omega^2}{c^5} \quad (81.13)$$

A comparison of expressions (81.12) and (81.13) shows that

$$P_m : P_p = \left(\frac{v}{c} \right)^2 : 1$$

Recall that we have calculated the fields of radiated waves for the non-relativistic case (i.e. for $v \ll c$). Consequently, in the case we have studied, the intensity of magnetic-dipole radiation is much lower than that of electric-dipole radiation.

According to definition (43.13), the quadrupole moment (and, therefore, the vector Q) is a quantity of the order of er'^2 , i.e.

$$Q \sim el^2 \cos^2 \omega t,$$

$$\dot{Q} \sim el^2 \omega \cos \omega t \sin \omega t \sim el^2 \omega \sin 2\omega t$$

$$\ddot{Q} \sim el^2 \omega^2 \cos 2\omega t, \quad \ddot{\dot{Q}} \sim e^2 l^2 \omega^3 \sin 2\omega t$$

Hence,

$$P_Q \sim \frac{\ddot{Q}^2}{c^5} \sim \frac{e^2 l^4 \omega^6}{c^5} \sim \frac{e^2 v^4 \omega^2}{c^5} \quad (81.14)$$

Quadrupole radiation thus has an intensity of the same order of magnitude as magnetic-dipole radiation.

APPENDICES

I. Lagrange's Equations for a Holonomic System with Ideal Non-Stationary Constraints

For non-stationary constraints, conditions (4.2) have the form

$$f_l(x_1, x_2, \dots, x_n, t) = 0 \quad (l = 1, 2, \dots, r) \quad (\text{I.1})$$

Accordingly, the time also enters functions (4.3):

$$x_i = x_i(q_1, q_2, \dots, q_s, t) \quad (i = 1, 2, \dots, n) \quad (\text{I.2})$$

Another term appears in formulas (4.4) and (4.5):

$$\dot{x}_i = \frac{\partial x_i}{\partial t} + \sum_l \frac{\partial x_i}{\partial q_l} \dot{q}_l \quad (\text{I.3})$$

From (I.3), we obtain the relation

$$\frac{\dot{\partial x_i}}{\partial \dot{q}_l} = \frac{\partial x_i}{\partial q_l} \quad (\text{I.4})$$

coinciding with (4.6). Formulas (4.7) also remain unchanged:

$$\frac{\partial x_i}{\partial \dot{q}_h} = 0 \quad (\text{I.5})$$

Since the quantities $\partial x_i / \partial q_h$ contain not only q_h , but also t , expression (4.8) becomes somewhat more involved:

$$\begin{aligned} \frac{d}{dt} \frac{\partial x_i}{\partial q_h} &= \frac{\partial}{\partial t} \left(\frac{\partial x_i}{\partial q_h} \right) + \sum_l \frac{\partial}{\partial q_l} \left(\frac{\partial x_i}{\partial q_h} \right) \dot{q}_l \\ &= \frac{\partial^2 x_i}{\partial t \partial q_h} + \sum_l \frac{\partial^2 x_i}{\partial q_l \partial q_h} \dot{q}_l \end{aligned} \quad (\text{I.6})$$

Differentiation of expression (I.3) with respect to q_h yields

$$\frac{\partial \dot{x}_i}{\partial q_h} = \frac{\partial}{\partial q_h} \left(\frac{\partial x_i}{\partial t} \right) + \sum_l \frac{\partial}{\partial q_h} \left(\frac{\partial x_i}{\partial q_l} \right) \dot{q}_l = \frac{\partial^2 x_i}{\partial q_h \partial t} + \sum_l \frac{\partial^2 x_i}{\partial q_h \partial q_l} \dot{q}_l$$

Comparing this expression with (I.6), we arrive at the relation

$$\frac{\partial \dot{x}_i}{\partial q_h} = \frac{d}{dt} \frac{\partial x_i}{\partial q_h} \quad (\text{I.7})$$

coinciding with (4.9).

Let us multiply the equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = R_i + F_i^* \quad (i = 1, 2, \dots, n)$$

[see (4.1)] by $\partial x_i / \partial q_h$ and summate them over i :

$$\sum_i \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right) \frac{\partial x_i}{\partial q_h} - \sum_i \frac{\partial L}{\partial x_i} \frac{\partial x_i}{\partial q_h} = \sum_i R_i \frac{\partial x_i}{\partial q_h} + \sum_i F_i^* \frac{\partial x_i}{\partial q_h} \quad (\text{I.8})$$

Since relations (4.9), (4.6), and (4.7), used in Sec. 4 in the transformation of the left-hand side of this equation, remained unchanged [see (I.7), (I.4) and (I.5)], the result will also remain unchanged. Consequently, the left-hand side of (I.8) can be written as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial q_h}$$

[see the paragraph preceding formula (4.15)].

Before starting to consider the right-hand side of formula (I.8), let us discuss the following matter. With stationary constraints, $x_i = x_i(q_h)$, and the increment of the coordinate x_i during the time dt will be

$$dx_i = \sum_h \frac{\partial x_i}{\partial q_h} dq_h \quad (\text{I.9})$$

where dq_h are the increments of the generalized coordinates during the time dt . With non-stationary constraints, $x_i = x_i(q_h, t)$, and the increment of x_i during the time dt is

$$dx_i = \frac{\partial x_i}{\partial t} dt + \sum_h \frac{\partial x_i}{\partial q_h} dq_h \quad (\text{I.10})$$

where dq_h are again the increments of the coordinates q_h during the time dt .

If the constraints were to suddenly stop changing, $\partial x_i / \partial t$ would vanish, and formula (I.10) would coincide with (I.9). Consequently, the second term in (I.10) is the imaginary displacement of the system that would be obtained with "frozen" constraints. It is called the **virtual** (or possible) displacement and is designated by δx_i . Hence, the true displacement dx_i can be written as the sum of the virtual displacement δx_i and an addend equal to $(\partial x_i / \partial t) dt$:

$$dx_i = \delta x_i + \frac{\partial x_i}{\partial t} dt \quad (\text{I.11})$$

With stationary constraints, the virtual displacement coincides with the true one.

Now let us consider the first term on the right-hand side of formula (I.8). Multiplying it by dq_k , we obtain

$$\left(\sum_i R_i \frac{\partial x_i}{\partial q_k} \right) dq_k = \sum_i R_i \left(\frac{\partial x_i}{\partial q_k} dq_k \right) = \sum_i R_i \delta x_i \quad (\text{I.12})$$

where δx_i is the virtual increment of the coordinate x_i appearing when only one generalized coordinate q_k changes.

The work of the reactions R_i with fixed non-stationary constraints (as with stationary constraints) is zero. Therefore, $\sum_i R_i \delta x_i = 0$, and since $dq_k \neq 0$, it follows from (I.12) that

$$\sum_i R_i \frac{\partial x_i}{\partial q_k} = 0 \quad (\text{I.13})$$

We must note that the true work of the reactions of non-stationary constraints according to (I.11) and (I.13) is

$$\sum_i R_i dx_i = \sum_i R_i \delta x_i + \sum_i R_i \frac{\partial x_i}{\partial t} dt = \sum_i R_i \frac{\partial x_i}{\partial t} dt$$

and, generally speaking, is non-zero.

Hence, according to (I.13), the first term on the right-hand side of formula (I.8) is zero—the reactions of the constraints have again vanished from the equations. As regards the second term, by definition it is the generalized force Q_k^* [see formula (4.12)]. Consequently, for non-stationary constraints too, we have arrived at Lagrange's equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k^*$$

II. Euler's Theorem for Homogeneous Functions

A function of any number of variables is called a homogeneous function of these variables of the degree m if upon multiplying all the variables by the arbitrary quantity α , the function is multiplied by α^m , i.e.

$$f(\alpha x_1, \alpha x_2, \dots, \alpha x_n) \equiv \alpha^m f(x_1, x_2, \dots, x_n)$$

Differentiation of this identity with respect to α yields

$$\sum_{i=1}^n \frac{\partial f}{\partial (\alpha x_i)} x_i = m \alpha^{m-1} f(x_1, x_2, \dots, x_n)$$

Assuming that $\alpha = 1$, we obtain

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i} x_i = m f(x_1, x_2, \dots, x_n)$$

We have proved Euler's theorem, which states that *the sum of the products of partial derivatives of a homogeneous function and the corresponding variables equals the product of the function itself and the degree of its homogeneity.*

III. Some Information from the Calculus of Variations

1. **Functional.** If to each number x belonging to a certain class there corresponds another number y , it is general knowledge that we have to do with the function $y = y(x)$.

If to each function $y(x)$ belonging to a certain class of functions there corresponds a certain number Φ , the functional $\Phi[y(x)]$ is said to be set.

For purposes of clarity, we shall sometimes speak of a curve instead of a function.

Hence, a function establishes the correspondence:

number \rightarrow number

whereas a functional establishes the correspondence:

function (or curve) \rightarrow number

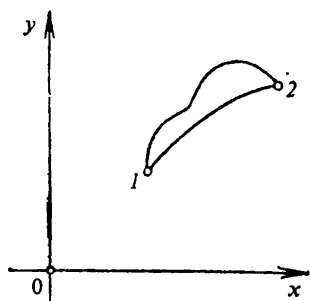


Fig. III.1.

Consequently, when dealing with a functional, the role of the argument is played by a function (or a curve).

Let us explain what has been said above by means of the following example. Assume that we are given two fixed points 1 and 2 in the plane x, y (Fig. III.1). The distance l_{12} between the points measured along the curve joining them is a functional. To find an analytical expression relating the quantity l_{12} to the function $y = y(x)$ describing the curve, we shall take into account that the element dl of the curve is related to dx and dy by the expression $dl^2 = dx^2 + dy^2$.

Writing dy as $y'(x) dx$, we obtain $dl = \sqrt{dx^2 + [y'(x)]^2 dx^2} = \sqrt{1 + [y'(x)]^2} dx$. Finally, integrating and designating l_{12} by $\Phi[y(x)]$, we arrive at the expression

$$\Phi[y(x)] = \int_1^2 \sqrt{1 + [y'(x)]^2} dx \quad (\text{III.1})$$

Taking different curves, i.e. different functions $y(x)$, we shall obtain different numbers Φ .

In the same way as there are functions not of one, but of several variables, there are functionals depending on several functions: $y_1(x), y_2(x), \dots, y_s(x)$ ¹.

If a functional satisfies the conditions

$$\Phi[cy(x)] = c\Phi[y(x)] \quad (c = \text{constant}) \quad (\text{III.2})$$

$$\Phi[y_1(x) + y_2(x)] = \Phi[y_1(x)] + \Phi[y_2(x)]$$

it is called **linear**. We shall denote a linear functional by the symbol Φ_{lin} , or F_{lin} , etc.

Functionals depending on several functions may be linear with respect to some of them and non-linear with respect to the others. In this case, for example, the symbol

$$\Phi_{\text{lin } y_2}[y_1(x), y_2(x), \dots]$$

will stand for a functional that is linear relative to the function $y_2(x)$.

It is the task of the calculus of variations to work out methods for finding the extremal (i.e. maximum, minimum, or stationary) values of functionals. This task is in many aspects similar to the task of finding the extrema of conventional functions.

2. Variation of a Functional. Let us select an arbitrary function $\tilde{y}(x)$ from a class of functions being considered (we have used the tilde sign to distinguish the selected function from the remaining functions of the given class). Now let us select another function $y(x)$ from the same class. The difference of these two functions is called the **variation** of the function $\tilde{y}(x)$. The variation is designated by the symbol $\delta y(x)$ or simply δy . The variation of a function is thus determined by the following expression:

$$\delta y = y(x) - \tilde{y}(x) \quad (\text{III.3})$$

The variation of a function is similar to the increment Δx (or dx) of the argument of an ordinary function: $\Delta x = x - \tilde{x}$.

The variation δy of the function $\tilde{y}(x)$ is evidently a function of x . Differentiating this function with respect to x , we find in accordance with (III.3) that

$$(\delta y)' = y'(x) - \tilde{y}'(x)$$

The right-hand side of this expression is the variation of the function $\tilde{y}'(x)$. Consequently, we arrive at the relation

$$(\delta y)' = \delta y' \quad (\text{III.4})$$

(the derivative of a variation is the variation of the derivative).

¹ There are also functionals depending on the functions of several variables x_1, x_2, \dots, x_n . We shall not need such functionals, however, and shall not consider them.

Let us determine what quantity in the calculus of variations corresponds to the differential of a conventional function. Recall that if a function is continuous¹, its increment Δy equal to $y(x) - \tilde{y}(x)$ can be written as the sum of a term that is linear with respect to Δx and an infinitesimal of an order higher than the first one relative to Δx :

$$\Delta y = y'(\tilde{x}) \Delta x + \varepsilon \Delta x$$

where ε is a quantity that vanishes together with Δx (in other words, $\lim_{\Delta x \rightarrow 0} \varepsilon = 0$). The first term in this expression is called the differential of the function. Hence, a differential of a function is defined to be that part of the increment of the function that is linear relative to Δx .

For the function $y = x$, the differential coincides with the increment. Consequently, $\Delta x = dx$, so that the expression for the differential of a function can be written as

$$dy = y'(x) \Delta x \text{ or } dy = y'(x) dx \quad (\text{III.5})$$

If a small change in a functional corresponds to a small change in a function (to a small δy), the functional is called **continuous**. A quantity similar to the differential of a conventional function can be introduced for continuous functionals.

The increment of a functional $\Delta \Phi = \Phi[y(x) + \delta y] - \Phi[y(x)]$ is a quantity depending on two functions: $y(x)$ and δy . Therefore, $\Delta \Phi$ is also a functional. This functional, generally speaking, will be non-linear. If $\Delta \Phi[y(x)]$ can be written as the sum of the functional $F_{\text{lin}, \delta y}[y(x), \delta y]$ that is linear relative to δy and an infinitesimal of an order higher than the first one with respect to $|\delta y|_{\text{max}}$ (the maximum value of the magnitude of the function δy), the main part of the increment of a functional linear relative to δy is called the **variation of the functional**. Hence,

$$\Delta \Phi[y(x)] = \delta \Phi[y(x)] + \varepsilon |\delta y|_{\text{max}} \quad (\text{III.6})$$

where

$$\delta \Phi[y(x)] = F_{\text{lin}, \delta y}[y(x), \delta y] \quad (\text{III.7})$$

is the variation of the functional, and ε is a quantity vanishing together with $|\delta y|_{\text{max}}$.

The variation of a functional is the analogue of the differential of a function determined by expression (III.5).

¹ More strictly, the function ought to be not only continuous, but also differentiable. As a rule, however, the functions considered in physics are also differentiable if they are continuous. Although, generally speaking, continuous functions are known in mathematics that are not differentiable.

The definition (III.7) can easily be generalized for functionals depending on several functions:

$$\begin{aligned} \delta\Phi [y_1(x), y_2(x), \dots, y_s(x)] \\ = F_{11n, \delta y_1, \delta y_2, \dots, \delta y_s} [y_1(x), y_2(x), \dots, y_s(x), \delta y_1, \delta y_2, \dots, \delta y_s] \end{aligned} \quad (\text{III.8})$$

As an illustration, let us find the increment of the functional

$$\Phi[y(x)] = \int_a^b [y(x)]^2 dx \quad (\text{III.9})$$

This increment is

$$\begin{aligned} \Delta\Phi = \Phi[y(x) + \delta y] - \Phi[y(x)] &= \int_a^b [y(x) + \delta y]^2 dx - \int_a^b [y(x)]^2 dx \\ &= \int_a^b [2y(x)\delta y + (\delta y)^2] dx = \int_a^b 2y(x)\delta y dx + \int_a^b (\delta y)^2 dx \end{aligned}$$

The integral $\int_a^b (\delta y)^2 dx$ does not exceed the quantity $|\delta y|_{\max}^2 (b - a) = \{|\delta y|_{\max} (b - a)\} |\delta y|_{\max}$, where $|\delta y|_{\max}$ is the maximum value of the magnitude of the function δy within the interval $a \leq x \leq b$. The expression in braces vanishes together with $|\delta y|_{\max}$.

Consequently, $\int_a^b (\delta y)^2 dx$ can be written as $\varepsilon |\delta y|_{\max}$, where $\varepsilon \rightarrow 0$ when $|\delta y|_{\max} \rightarrow 0$. Hence,

$$\Delta\Phi = \int_a^b 2y(x)\delta y dx + \varepsilon |\delta y|_{\max}$$

[compare with (III.6)]. The first term in this expression is the functional depending on the functions $y(x)$ and δy :

$$F_{11n, \delta y} [y(x), \delta y] = \int_a^b 2y(x)\delta y dx \quad (\text{III.10})$$

It is easy to verify that this functional is linear with respect to δy^1 [see the conditions (III.2)]. Hence, expression (III.10) gives the variation of the functional (III.9).

¹ The given functional is also linear with respect to $y(x)$, but this is of no significance—only the linearity with respect to δy is important.

3. Necessary Condition for the Extremum of a Functional. Let us again begin with similar concepts from calculus. The function $f(x_1, x_2, \dots, x_n)$ is said to have an extremum at the point $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ if the increment of this function

$$\Delta f = f(x_1, x_2, \dots, x_n) - f(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$$

has the same sign for all the points (x_1, x_2, \dots, x_n) belonging to the vicinity of the point $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$. There is a maximum at a given point when $\Delta f \leq 0$, and a minimum when $\Delta f \geq 0$.

It is proved in calculus that a necessary condition for the existence of an extremum at a point is the equality to zero of the differential of the function at this point:

$$df = \sum_{h=1}^n \frac{\partial f}{\partial x_h} dx_h = 0 \quad (\text{III.11})$$

The functional $\Phi[y(x)]$ is similarly said to reach an extremum at $y = \tilde{y}(x)$ if the increment of the functional

$$\Delta\Phi = \Phi[y(x)] - \Phi[\tilde{y}(x)]$$

has the same sign for all the curves $y(x)$ sufficiently close to the curve $\tilde{y}(x)$. When $\Delta\Phi \leq 0$, a maximum of the functional is observed, and when $\Delta\Phi \geq 0$, a minimum.

Let us find the condition necessary for the functional $\Phi[y(x)]$ to reach an extremum at $y = \tilde{y}(x)$. For clarity, we shall consider the case of a maximum. If $\Phi[y(x)]$ reaches a maximum at $y = \tilde{y}(x)$, this signifies that

$$\Phi[\tilde{y}(x) + \delta y] - \Phi[\tilde{y}(x)] \leq 0 \quad (\text{III.12})$$

for all $\delta y = \delta y(x)$ for which $|\delta y|_{\max}$ is sufficiently small. By (III.7) and (III.8)

$$\begin{aligned} \Delta\Phi &= \Phi[\tilde{y}(x) + \delta y] - \Phi[\tilde{y}(x)] \\ &= F_{11n, \delta y}[\tilde{y}(x), \delta y] + \varepsilon |\delta y|_{\max} \end{aligned} \quad (\text{III.13})$$

Let us separate from all the possible variations δy those that can be represented in the form $\delta y = \alpha \delta y_0$, where δy_0 is a fixed sufficiently small variation, and α is a varying algebraic quantity. Introducing this variation into (III.13) and taking into account that owing to linearity $F[\tilde{y}(x), \alpha \delta y_0] = \alpha F[\tilde{y}(x), \delta y_0]$, we can write

$$\Delta\Phi = \alpha F[\tilde{y}(x), \delta y_0] + \varepsilon \alpha |\delta y_0|_{\max}$$

In the last expression, $F[\tilde{y}(x), \delta y_0]$ is simply a number. If this number is non-zero, at sufficiently small α 's the sign of $\Delta\Phi$ will

be determined by that of the expression $\alpha F[\tilde{y}(x), \delta y_0]$ (the term $\varepsilon \alpha |\delta y_0|_{\max}$ diminishes much more rapidly than the first term¹), and this expression will change its sign together with α (which may be either positive or negative). Hence, for the condition (III.12) to be satisfied, the quantity $F[\tilde{y}(x), \delta y_0]$ must equal zero. The condition (III.12) must be satisfied for all sufficiently small variations δy without any exception. We have found, however, that if $\delta\Phi \neq 0$ for even part of the variations having the form $\alpha\delta y_0$, the above condition is not observed. We can therefore state the following: *for the functional $\Phi[y(x)]$ to achieve a maximum at $y = \tilde{y}(x)$, its variation (if it exists) must vanish at $y = \tilde{y}(x)$:*

$$\delta\Phi = F_{\text{lin}, \delta y}[\tilde{y}(x), \delta y] \equiv 0 \quad (\text{III.14})$$

[The identity sign underlines the circumstance that the condition (III.14) must be satisfied for all the δy 's.]

It can be seen that by repeating our reasoning for the minimum of a functional, we shall arrive at the same conclusion. Consequently, formula (III.14) expresses the condition necessary not only for a maximum, but also for a minimum, i.e. for an extremum in general. This formula is an analogue of formula (III.11).

4. A Simple Problem in the Calculus of Variations. Let us find the extremum of a functional having the form

$$\Phi[y(x)] = \int_{x_1}^{x_2} f[x, y(x), y'(x)] dx \quad (\text{III.15})$$

The boundary points of the allowable curves are assumed to be fixed—for all the allowable curves

$$y(x_1) = y_1 \text{ and } y(x_2) = y_2 \quad (\text{III.16})$$

The increment of the functional is

$$\Delta\Phi = \int_{x_1}^{x_2} f[x, y + \delta y, y' + \delta y'] dx - \int_{x_1}^{x_2} f[x, y, y'] dx$$

We expand the integrand of the first integral in powers of the small quantities δy and $\delta y'$. The result is

$$\begin{aligned} \Delta\Phi = \int_{x_1}^{x_2} \left\{ f[x, y, y'] + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right. \\ \left. + \varepsilon(\delta y, \delta y') \right\} dx - \int_{x_1}^{x_2} f[x, y, y'] dx \end{aligned}$$

¹ The condition $\varepsilon \rightarrow 0$ for $|\delta y|_{\max} \rightarrow 0$ in the given case acquires the form $\lim_{\alpha \rightarrow 0} \varepsilon = 0$.

where $\varepsilon(\delta y, \delta y')$ combines the terms of an order higher than the first one relative to the quantities δy and $\delta y'$. This expression is simplified as follows:

$$\Delta\Phi = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right\} dx + \int_{x_1}^{x_2} \varepsilon(\delta y, \delta y') dx \quad (\text{III.17})$$

Let us integrate the second term in the first integral by parts. For this purpose, we write it as

$$J = \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \delta y' dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} (\delta y)' dx$$

[remember that $\delta y' = (\delta y)'$, see formula (III.4)]. Designating $\partial f / \partial y'$ by u , and $(\delta y)' dx$ by dv and taking advantage of the formula $\int u dv = uv - \int v du$, we obtain

$$J = \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} (\delta y)' dx = \frac{\partial f}{\partial y'} \delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \delta y dx$$

Since the boundary points of the allowed curves are fixed, the variation δy at these points must vanish: $\delta y(x_1) = 0$, and $\delta y(x_2) = 0$. Therefore, the first term on the right-hand side vanishes, and

$$J = - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \delta y dx$$

Let us introduce the found value of J into formula (III.17), factoring out δy :

$$\Delta\Phi = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right\} \delta y dx + \int_{x_1}^{x_2} \varepsilon(\delta y, \delta y') dx$$

The main part of $\Delta\Phi$ is formed by the first integral that is a functional linear with respect to δy . By definition, this integral is the variation of the functional Φ . Hence, the variation of the functional (III.15) is

$$\delta\Phi = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right\} \delta y dx \quad (\text{III.18})$$

and the condition (III.14) for an extremum will be written as

$$\int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right\} \delta y dx \equiv 0$$

The identity obtained must be observed for any sufficiently small functions $\delta y = \delta y(x)$. This is possible only provided that the integrand in braces vanishes:

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0 \quad (\text{III.19})$$

This equation is known as Euler's equation. It is the condition for an extremum of functionals of the form given by (III.15). The curves $y = y(x, C_1, C_2)$, which are solutions of this equation, are called **extremals** (C_1 and C_2 are integration constants).

We must note that the addition to the integrand in (III.15) of the total derivative with respect to x of any function $\psi(y, x)$ does not change the conditions of the extremum (III.19). Indeed, this term after integration yields the quantity

$$\int_{x_1}^{x_2} \frac{d\psi}{dx} dx = \psi(y_2, x_2) - \psi(y_1, x_1)$$

whose variation is zero [according to the condition (III.16), the curves do not vary at their ends]. Hence, the addition of $d\psi/dx$ changes only the value of the extremum of a functional, but does not affect the form of the function $y(x)$ at which this extremum is reached.

Let us use formula (III.19) to find the extremum of the functional (III.1). In this case, $f(x, y, y') = \sqrt{1 + [y'(x)]^2}$. Hence,

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = -\frac{y'(x)}{\sqrt{1 + [y'(x)]^2}}$$

and Eq. (III.19) becomes

$$\frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} = \left(\frac{1}{\sqrt{1 + y'^2}} - \frac{y'^2}{(1 + y'^2)^{3/2}} \right) y'' = 0$$

A function for which $y'' = 0$ and $y' = a$ will be the solution of this differential equation. Consequently, the function itself is a linear one, $y = ax + b$, whose coefficients must be chosen so as to satisfy the conditions (III.16). Therefore, functional (III.1) reaches an extremum (in this case, evidently, a minimum) if we presume that $y(x)$ is a straight line joining the points 1 and 2 (see Fig. III.1).

5. Extremum of Functionals Depending on Several Functions. Consider a functional of the form

$$\Phi[y_1, y_2, \dots, y_s] = \int_{x_1}^{x_2} f[x, y_1, y_2, \dots, y_s, y'_1, y'_2, \dots, y'_s] dx \quad (\text{III.20})$$

where $y_k = y_k(x)$ ($k = 1, 2, \dots, s$) are curves fixed at boundary points, i.e. satisfying the boundary conditions

$$y_k(x_1) = y_{k1}, \quad y_k(x_2) = y_{k2} \quad (k = 1, 2, \dots, s) \quad (\text{III.21})$$

In accordance with what has been said in paragraph 4, the variation of the functional (III.20) is determined by the expression

$$\delta\Phi = \int_{x_1}^{x_2} \sum_{k=1}^s \left(\frac{\partial f}{\partial y_k} \delta y_k + \frac{\partial f}{\partial y'_k} \delta y'_k \right) dx \quad (\text{III.22})$$

where δy_k are sufficiently small functions of x vanishing at the boundary points

$$\delta y_k(x_1) = 0, \quad \delta y_k(x_2) = 0 \quad (\text{III.23})$$

It is necessary to find such a set of functions $\tilde{y}_k(x)$ satisfying the conditions (III.21) at which the functional (III.20) reaches an extremum. A necessary condition for an extremum is the vanishing of the variation of the functional (III.22).

Integrating by parts each of the s addends of the form

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'_k} \delta y'_k dx$$

in formula (III.22), we can bring it to the form

$$-\int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) \delta y_k dx$$

Performing such a replacement in (III.22) and equating to zero the expression obtained for $\delta\Phi$, we arrive at the necessary condition for an extremum:

$$\int_{x_1}^{x_2} \sum_{k=1}^s \left(\frac{\partial f}{\partial y_k} - \frac{d}{dx} \frac{\partial f}{\partial y'_k} \right) \delta y_k dx \equiv 0 \quad (\text{III.24})$$

This identity must be obeyed for any sufficiently small functions $\delta y_k = \delta y_k(x)$ selected independently of one another. This is possible only provided that all s expressions in parentheses inside the sum are zero. We have thus arrived at a system of Euler's equations:

$$\frac{\partial f}{\partial y_k} - \frac{d}{dx} \frac{\partial f}{\partial y'_k} = 0 \quad (k = 1, 2, \dots, s) \quad (\text{III.25})$$

The set of functions $y_k = \tilde{y}_k(x)$ satisfying these equations and the boundary conditions (III.21) when substituted into the functional (III.20) will give its extremum.

IV. Conics

Conics (conic sections) are defined to be the lines of intersection of a circular cone with a plane. Depending on the orientation of the plane relative to the axis of the cone, these lines are either an ellipse (circle), a hyperbola, or a parabola.

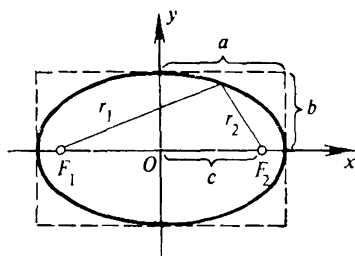


Fig. IV.1.

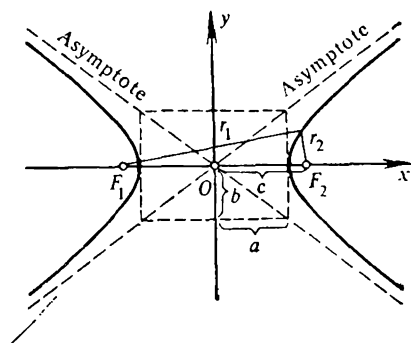


Fig. IV.2.

An ellipse (Fig. IV.1) is defined to be the locus of points, the sum of whose distances from two fixed points F_1 and F_2 called the foci is a constant quantity:

$$r_1 + r_2 = 2a \quad (\text{IV.1})$$

The canonical equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{IV.2})$$

where a and b are the major and minor semiaxes of the ellipse. The quantity

$$e = \frac{c}{a} \quad (\text{IV.3})$$

where c is half the distance between the foci, is called the eccentricity of the ellipse. When $e = 0$, an ellipse degenerates into a circle. The quantities a , b and c are related by the expression

$$b^2 = a^2 - c^2 \quad (\text{IV.4})$$

A hyperbola (Fig. IV.2) is defined to be the locus of points, the magnitude of the difference of whose distances from two fixed points F_1 and F_2 called the foci is a constant quantity:

$$|r_1 - r_2| = 2a \quad (\text{IV.5})$$

A hyperbola has two symmetric branches. The canonical equation of a hyperbola is as follows:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\text{VI.6})$$

where a and b are the real and imaginary semiaxes of the hyperbola. The quantities a and b are related to c (half the distance between the foci) by the expression

$$b^2 = c^2 - a^2 \quad (\text{IV.7})$$

The eccentricity of a hyperbola is determined by the same formula (IV.3) as that of an ellipse. It can be seen that for an ellipse $e < 1$, and for a hyperbola $e > 1$.

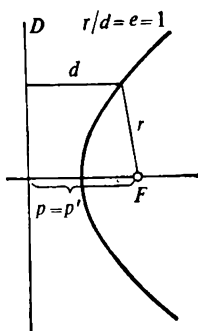


Fig. IV.3.

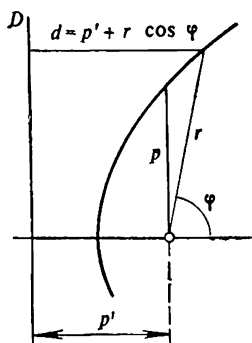


Fig. IV.4.

A parabola (Fig. IV.3) is defined to be the locus of points whose distance r from a fixed point F (the focus) equals the distance d from the fixed straight line D called the directrix of the parabola: $r = d$. The canonical equation of a parabola is

$$y^2 = 2px \quad (\text{IV.8})$$

where p is the parameter of the parabola, equal to the distance from the focus to the directrix (the x -axis is directed along the axis of symmetry of the parabola, and the origin of coordinates coincides with the apex of the parabola). As we shall see below, the eccentricity of a parabola should be taken equal to unity.

Any conic can be determined as the locus of points for which the ratio between the distance r (Fig. IV.4) to the point F (called the focus) and the distance d to the straight line D (called the directrix) is a constant quantity e (called the eccentricity of the curve):

$$\frac{r}{d} = e \quad (\text{IV.9})$$

The relevant conic is obtained depending on the value of the eccentricity e (Table IV.1).

Table IV.1

| Value of e | Kind of curve |
|--------------|---------------|
| < 1 | Ellipse |
| $= 1$ | Parabola |
| > 1 | Hyperbola |

An ellipse and a hyperbola each has two foci and two directrices. The condition (IV.9) is observed for each of the foci and the directrix corresponding to it. Figure IV.5 shows the foci and directrices of an ellipse (a) and a hyperbola (b) (compare with Fig. IV.3 for a parabola).

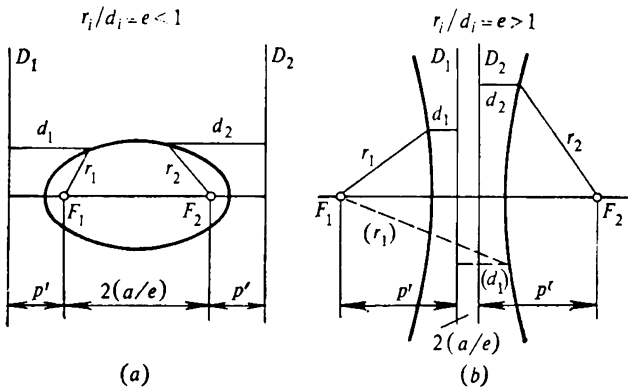


Fig. IV.5.

The distance p' from a focus to a directrix is called the parameter of the relevant curve. The quantity

$$p = p'e \quad (\text{IV.10})$$

is known as the focal parameter. It is a simple matter to see that it equals half the chord passing through a focus and parallel to the directrix (Fig. IV.4). For a parabola, $p = p'$.

The distance between the directrices of an ellipse (hyperbola) equals $2(a/e)$, where a is the major (real) semiaxis of the curve.

Let us write the equation of a conic in polar coordinates, placing the origin of coordinates at one of the foci of the curve (Fig. IV.4). In accordance with (IV.9)

$$\frac{r}{d} = \frac{r}{p' + r \cos \varphi} = e$$

whence

$$r = \frac{p}{1 - e \cos \varphi} \quad (\text{IV.11})$$

[$p = p'e$; see (IV.10)].

Equation (IV.11) describes an ellipse (with the origin of coordinates at the point F_1 ; Fig. IV.5a), the right-hand branch of a hyperbola (with the origin of coordinates at the point F_2 ; Fig. IV.5b), and a parabola. We must note that the focus F_2 is the inner one for the right-hand branch of the hyperbola.

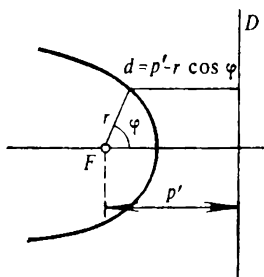


Fig. IV.6.

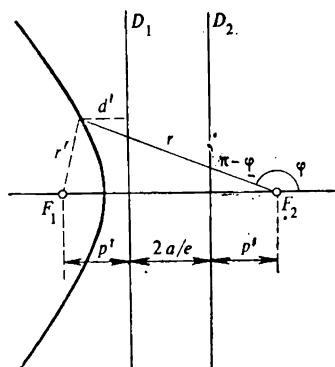


Fig. IV.7.

If we place the origin of coordinates at the right-hand focus of an ellipse (point F_2 in Fig. IV.5a), a glance at Fig. IV.6 shows that the equation of the ellipse is

$$\frac{r}{p' - r \cos \varphi} = e$$

or

$$r = \frac{p}{1 + e \cos \varphi} \quad (\text{IV.12})$$

($p = p'e$). The same equation describes the left-hand branch of a hyperbola (see Fig. IV.5b) provided that the origin of coordinates is placed at the point F_1 (at the inner focus with respect to this branch), and also a parabola that is the mirror (relative to D) image of the parabola depicted in Fig. IV.3. The focus F of such a parabola is to the left of the directrix D .

Let us find the equation of one of the branches of a hyperbola (say, the left one) provided that the origin of coordinates is at the external focus relative to this branch (at the point F_2 for the left

branch; Fig. IV.5b). In this case (Fig. IV.7),

$$d' = r \cos(\pi - \varphi) - 2 \frac{a}{e} - p' = -r \cos \varphi - 2 \frac{a}{e} - p'$$

According to the definition (IV.5) of a hyperbola, $r - r' = 2a$, whence

$$r' = r - 2a$$

Introducing the values d' and r' we have found into relation (IV.9), we arrive at the formula

$$\frac{r'}{d'} = \frac{r - 2a}{-r \cos \varphi - 2 \frac{a}{e} - p'} = e$$

After simple transformations, we obtain the required equation

$$r = \frac{-p}{1 + e \cos \varphi} \quad (\text{IV.13})$$

($p = p'e$). It must be borne in mind that for the left branch $\varphi > \pi/2$ (see Fig. IV.7), i.e. $\cos \varphi < 0$. In addition, for all points $|e \cos \varphi| > 1$ so that the values of r obtained by formula (IV.13) will be positive.

We invite our reader to convince himself that the similar equation for the right branch (the origin of coordinates is at the point F_1) is

$$r = \frac{-p}{1 - e \cos \varphi} \quad (\text{IV.14})$$

in this case, $\varphi < \pi/2$ so that $\cos \varphi > 0$. In addition, $e \cos \varphi > 1$ so that the values of r are positive.

V. Linear Differential Equations with Constant Coefficients

A linear differential equation of the n -th order with constant coefficients is defined to be an equation of the kind

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = f(x) \quad (\text{V.1})$$

s.e. an equation linear in the unknown function $y(x)$ and its derivatives (the a_i 's are constant quantities that may also be zero).

If the right-hand side of an equation identically equals zero [$f(x) \equiv 0$], the linear equation is called homogeneous, otherwise it is non-homogeneous. A homogeneous equation has the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = 0 \quad (\text{V.2})$$

The general solution of a differential equation is defined to be the multitude of solutions including all the particular solutions with no exceptions. The general solution of an n -th order differential equation contains n arbitrary constants (integration constants),

i.e. has the form

$$y = y(x, C_1, C_2, \dots, C_n) \quad (\text{V.3})$$

By giving the constants C_1, C_2, \dots, C_n definite values, we obtain a particular solution. The latter contains no arbitrary constants.

It is proved in the theory of linear differential equations that if y_1, y_2, \dots, y_n are linearly independent¹ solutions of the homogeneous equation (V.2), the general solution of this equation can be written as

$$y(x, C_1, C_2, \dots, C_n) = \sum_{i=1}^n C_i y_i(x) \quad (\text{V.4})$$

where C_1, C_2, \dots, C_n are arbitrary constants.

Let $\tilde{y}(x)$ be one of the particular solutions of the non-homogeneous equation (V.1) and $y(x)$ be the general solution of the same equation. If we introduce the notation $u(x) = y(x) - \tilde{y}(x)$, the general solution can be written as follows:

$$y(x, C_1, C_2, \dots, C_n) = u(x, C_1, C_2, \dots, C_n) + \tilde{y}(x) \quad (\text{V.5})$$

Let us substitute this function into Eq. (V.1) and group separately the terms of the kind $u^{(k)}$ and the terms of the kind $\tilde{y}^{(k)}$:

$$u^{(n)} + a_{n-1}u^{(n-1)} + \dots + a_1u' + a_0u + [\tilde{y}^{(n)} + a_{n-1}\tilde{y}^{(n-1)} + \dots + a_1\tilde{y}' + a_0\tilde{y}] = f(x)$$

The function $\tilde{y}(x)$ is a particular solution of the equation. Consequently, the expression in brackets equals the right-hand side of the equation. It follows that the function $u(x, C_1, C_2, \dots, C_n)$ satisfies the condition

$$u^{(n)} + a_{n-1}u^{(n-1)} + \dots + a_1u' + a_0u = 0$$

Hence, u is the general solution of the homogeneous equation (V.2) corresponding to the non-homogeneous equation (V.1), i.e. having the same coefficients a_k as Eq. (V.1).

The result we have obtained can be formulated as follows: *the general solution of a linear non-homogeneous equation equals the sum of the general solution of the corresponding homogeneous equation and a particular solution of the non-homogeneous equation:*

$$y(\text{gen., non-hom.}) = y(\text{gen., hom.}) + y(\text{part., non-hom.}) \quad (\text{V.6})$$

¹ The set of functions f_1, f_2, \dots, f_n is called linearly independent if an expression of the kind

$$\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_n f_n \equiv 0$$

is observed only provided that all the α_i 's vanish.

Linear homogeneous differential equations with constant coefficients are solved with the aid of the substitution

$$y(x) = e^{\lambda x} \quad (\text{V.7})$$

where λ is a constant. Differentiating this function m times ($m = 1, 2, \dots, n$), we obtain

$$y^{(m)} = \lambda^m e^{\lambda x} \quad (\text{V.8})$$

Introducing the values of the function (V.7) and its derivatives (V.8) into Eq. (V.2) and cancelling the non-zero factor $e^{\lambda x}$, we arrive at what is called a characteristic equation:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0 \quad (\text{V.9})$$

The roots of this equation are the values of λ at which the function (V.7) satisfies Eq. (V.2).

If all n roots of the characteristic equation are different (multiple, i.e. coinciding roots are absent), n particular solutions of the kind $e^{\lambda_i x}$ will be linearly independent. Consequently, in the absence of multiple roots, the general solution of Eq. (V.2) is as follows:

$$y = \sum_{i=1}^n C_i e^{\lambda_i x} \quad (\text{V.10})$$

(C_1, C_2, \dots, C_n are arbitrary constants).

It can be shown that when the characteristic equation (V.9) has multiple roots, p_μ linear independent particular solutions corresponding to the root λ_μ of multiplicity p_μ must be taken in the form

$$e^{\lambda_\mu x}, x e^{\lambda_\mu x}, x^2 e^{\lambda_\mu x}, \dots, x^{p_\mu-1} e^{\lambda_\mu x}$$

so that the contribution to the general solution corresponding to them equals the sum

$$\sum_{k=1}^{p_\mu} C_{\mu k} x^{k-1} e^{\lambda_\mu x}$$

Consequently, if the root λ_1 will be of the multiplicity p_1 , the root λ_2 of the multiplicity p_2 , \dots , λ_m of the multiplicity p_m (here $p_1 + p_2 + \dots + p_m = n$), the general solution can be written as

$$y = \sum_{\mu=1}^m \sum_{k=1}^{p_\mu} C_{\mu k} x^{k-1} e^{\lambda_\mu x} \quad (\text{V.11})$$

Let the coefficients a_k in Eq. (V.1) be real, and the function $f(x)$ be complex. Writing it in the form

$$f(x) = f_1(x) + i f_2(x)$$

we get the equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = f_1(x) + if_2(x) \quad (\text{V.12})$$

We shall seek the solution of this equation in the form

$$y(x) = y_1(x) + iy_2(x)$$

Substitution into (V.12) yields

$$\begin{aligned} (y_1^{(n)} + a_{n-1}y_1^{(n-1)} + \dots + a_1y_1' + a_0y_1) \\ + i(y_2^{(n)} + a_{n-1}y_2^{(n-1)} + \dots + a_1y_2' + a_0y_2) = \\ = f_1(x) + if_2(x) \quad (\text{V.13}) \end{aligned}$$

In complex numbers equal to one another, the real and imaginary parts are equal to one another independently. Hence, Eq. (V.13) breaks up into two independent equations of the form of (V.1). The right-hand side of one of them contains the function $f_1(x)$, and the function $y_1(x)$ is its solution. The right-hand side of the other equation contains the function $f_2(x)$, and the function $y_2(x)$ is its solution. This property of Eq. (V.13) is due to its linear nature. It allows us to use the following procedure that sometimes considerably facilitates calculations. Assume that the right-hand side of the equation (V.1) we are solving is real. We add an arbitrary imaginary function to it. After now finding the complex solution of the equation obtained, we take its real part. It will be a solution of the initial differential equation.

The following statement is obvious: if a linear homogeneous equation (V.2) with real coefficients has the complex solution $y(x) = y_1(x) + iy_2(x)$, each of the functions $y_1(x)$ and $y_2(x)$ separately is a solution of this equation.

VI. Vectors

1. Basic Definitions. Vectors are quantities defined by a numerical value (magnitude) and a direction and, in addition, are added geometrically (i.e. according to the triangle or parallelogram method). On a later page, we shall give a more general definition allowing us to extend the concept of a vector to an n -dimensional space.

The scalar product¹ of two vectors \mathbf{a} and \mathbf{b} is defined as the scalar quantity

$$\mathbf{a}\mathbf{b} = ab \cos(\mathbf{a}, \mathbf{b}) \quad (\text{VI.1})$$

A scalar product is commutative ($\mathbf{a}\mathbf{b} = \mathbf{b}\mathbf{a}$) and distributive $\{\mathbf{a}(\mathbf{b}_1 + \mathbf{b}_2 + \dots) = \mathbf{a}\mathbf{b}_1 + \mathbf{a}\mathbf{b}_2 + \dots\}$, but is not associative $\{\mathbf{a}(\mathbf{b}\mathbf{c}) \neq (\mathbf{a}\mathbf{b})\mathbf{c}\}$.

¹ Another way of writing a scalar product in addition to the one we use is $\mathbf{a} \cdot \mathbf{b}$, which explains the name dot product sometimes used for it.

A vector product is defined as the vector ¹

$$[\mathbf{a}\mathbf{b}] = ab \sin(\mathbf{a}, \mathbf{b}) \cdot \mathbf{n} \quad (\text{VI.2})$$

where \mathbf{n} is the unit vector of a normal to the plane containing the vectors \mathbf{a} and \mathbf{b} , the sequence $\mathbf{a}, \mathbf{b}, \mathbf{n}$ forming a right-handed system.

A vector product is not commutative ($[\mathbf{a}\mathbf{b}] \neq [\mathbf{b}\mathbf{a}]$), is distributive ($[\mathbf{a}, (\mathbf{b}_1 + \mathbf{b}_2 + \dots)] = [\mathbf{a}\mathbf{b}_1] + [\mathbf{a}\mathbf{b}_2] + \dots$), and is not associative ($[[\mathbf{a}\mathbf{b}], \mathbf{c}] \neq [\mathbf{a}, [\mathbf{b}\mathbf{c}]]$).

Let us consider a scalar triple product of three vectors: $\mathbf{a} [\mathbf{b}\mathbf{c}]$. Applying formulas (VI.1) and (VI.2), we obtain

$$\mathbf{a} [\mathbf{b}\mathbf{c}] = a \{bc \sin(\mathbf{b}, \mathbf{c})\} \cos(\mathbf{a}, \mathbf{n})$$

A glance at Fig. VI.1 shows that the expression we have obtained equals the volume of the parallelepiped constructed on the vectors being multiplied ². Indeed, $bc \sin(\mathbf{b}, \mathbf{c})$ gives the area of the base of the parallelepiped, and $a \cos(\mathbf{a}, \mathbf{n})$ gives its altitude.

We can also take a face whose sides form the vectors \mathbf{c} and \mathbf{a} or the vectors \mathbf{a} and \mathbf{b} as the base of the parallelepiped. In this case, the volume will be determined by the scalar triple products $\mathbf{b} [\mathbf{c}\mathbf{a}]$ and $\mathbf{c} [\mathbf{a}\mathbf{b}]$. Since the volume in all three cases is the same, we can write

$$\mathbf{a} [\mathbf{b}\mathbf{c}] = \mathbf{b} [\mathbf{c}\mathbf{a}] = \mathbf{c} [\mathbf{a}\mathbf{b}] \quad (\text{VI.3})$$

A scalar triple product thus allows a cyclic transposition of the factors, i.e. the substitution for each factor of the one following or preceding it. The vector \mathbf{c} is presumed to be followed by \mathbf{a} , and \mathbf{a} is assumed to be preceded by \mathbf{c} , which can be illustrated by the following diagram:

$$\begin{array}{c} \mathbf{a} \rightarrow \mathbf{b} \\ \swarrow \quad \searrow \\ \mathbf{c} \end{array} \quad (\text{VI.4})$$

We must note that in all three expressions of formula (VI.3) the vectors have the same sequence as in the diagram (VI.4). If we take the vectors in a sequence that is the opposite of what is shown in the diagram (VI.4), the scalar triple product will change its sign.

A vector triple product is defined to be the vector $[\mathbf{a} [\mathbf{b}\mathbf{c}]]$. We can prove (this will be done somewhat later) that

$$[\mathbf{a}, [\mathbf{b}\mathbf{c}]] = \mathbf{b} (\mathbf{a}\mathbf{c}) - \mathbf{c} (\mathbf{a}\mathbf{b}) \quad (\text{VI.5})$$

¹ Another way of writing a vector product is $\mathbf{a} \times \mathbf{b}$, which explains the name cross product sometimes used for it.

² The angle (\mathbf{a}, \mathbf{n}) is assumed to be acute. If this angle is obtuse, the scalar triple product equals the volume of the parallelepiped taken with a minus sign.

In accordance with the definitions (VI.2) and (VI.1), the square of the vector product of the vectors **a** and **b** can be transformed as follows:

$$[ab]^2 = a^2b^2 \sin^2(a, b) = a^2b^2 - a^2b^2 \cos^2(a, b) = a^2b^2 - (ab)^2$$

We have arrived at the formula

$$[ab]^2 = a^2b^2 - (ab)^2 \quad (\text{VI.6})$$

2. Formulas of Vector Algebra Expressed in Terms of Projections of Vectors onto the Coordinate Axes. All the above definitions and formulas do not depend on the choice of the coordinate system used for our consideration. If we set up a coordinate system (we shall consider only rectangular, i.e. Cartesian systems), each vector can be set by three numbers—its projections onto the coordinate axes¹. Consequently, the vector **a** is equivalent to the three numbers a_x, a_y, a_z , the vector **b**—to the numbers b_x, b_y, b_z , etc.

Knowing the projections of a vector onto the coordinate axes, we can find the vector itself. Denoting the unit vectors of the coordinate axes by the symbols $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$,² we can represent the vector as

$$\mathbf{a} = \mathbf{e}_x a_x + \mathbf{e}_y a_y + \mathbf{e}_z a_z \quad (\text{VI.7})$$

To obtain the possibility of writing the formulas in a compact form using the sum sign \sum , we shall use the symbol x_1 instead of the coordinate x , x_2 instead of y , and x_3 instead of z in the following. Similarly, we shall introduce the symbols $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for the unit vectors of the axes. The correspondence between the previous and the new symbols is shown below:

$$\left. \begin{array}{l} x \rightarrow x_1 \\ y \rightarrow x_2 \\ z \rightarrow x_3 \end{array} \right\} \quad (\text{VI.8})$$

$$\left. \begin{array}{l} \mathbf{i} = \mathbf{e}_x \rightarrow \mathbf{e}_1 \\ \mathbf{j} = \mathbf{e}_y \rightarrow \mathbf{e}_2 \\ \mathbf{k} = \mathbf{e}_z \rightarrow \mathbf{e}_3 \end{array} \right\} \quad (\text{VI.9})$$

In the new notation, formula (VI.7) can be written as

$$\mathbf{a} = \sum_k \mathbf{e}_k a_k \quad (\text{VI.10})$$

¹ We are treating free vectors, for which the point of application of the vector and the straight line along which it is directed are not fixed.

² These unit vectors are also designated by the symbols $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The notation we have adopted, however, as will be seen from the following treatment, has an undoubted advantage.

(unless otherwise indicated, we shall always assume that the subscript over which summation is performed—the dummy index—runs through the values 1, 2, 3).

The triplet of vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ forms the basis of a coordinate system. The setting of these vectors completely determines the system. Since $\mathbf{e}_1, \mathbf{e}_2$ and \mathbf{e}_3 are mutually perpendicular, and their magnitudes equal unity, it can be seen from formula (VI.1) that

$$\mathbf{e}_i \mathbf{e}_k = \delta_{ik} \quad (\text{VI.11})$$

where δ_{ik} is the Kronecker symbol determined as follows:

$$\delta_{ik} = \begin{cases} 1 & \text{when } i = k \\ 0 & \text{when } i \neq k \end{cases} \quad (\text{VI.12})$$

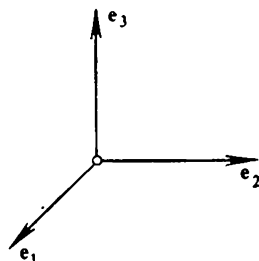


Fig. VI.2.

It must be noted that $\delta_{ik} = \delta_{ki}$ (it will be shown in Appendix X that the set of the quantities δ_{ik} forms a symmetric second-rank tensor).

Examination of formula (VI.2) reveals that (Fig. VI.2)

$$[\mathbf{e}_1 \mathbf{e}_2] = \mathbf{e}_3 \quad (\text{VI.13})$$

$$[\mathbf{e}_2 \mathbf{e}_3] = \mathbf{e}_1$$

$$[\mathbf{e}_3 \mathbf{e}_1] = \mathbf{e}_2$$

Each of the relations (VI.13) can be obtained from the preceding (or following) one by a cyclic transposition of the subscripts according to the diagram

$$\begin{array}{c} 1 \rightarrow 2 \\ \swarrow \quad \searrow \\ 3 \end{array} \quad (\text{VI.14})$$

Let us introduce the symbol¹

$$\varepsilon_{ikl} \quad (\text{VI.15})$$

standing for a set of 27 numbers that are determined by the following rules:

(1) if the values of at least two subscripts coincide, we have $\varepsilon_{ikl} = 0$ (for instance, $\varepsilon_{111} = \varepsilon_{222} = \varepsilon_{333} = \varepsilon_{221} = \varepsilon_{112} = \varepsilon_{331} = \varepsilon_{113} = \varepsilon_{223} = \varepsilon_{332} = 0$);

(2) if all the subscripts are different and form a cyclic transposition of the sequence 1, 2, 3, we have $\varepsilon_{ikl} = 1$ ($\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$);

(3) if all the subscripts are different and form a cyclic transposition of the sequence 3, 2, 1, we have $\varepsilon_{ikl} = -1$ ($\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$).

¹ It is sometimes called the Kronecker skew-symmetric symbol. It will be shown in Appendix X that the set of quantities ε_{ikl} forms an absolutely anti-symmetric third-rank tensor.

Hence, of the 27 values of ε_{ikl} , 21 are zero, 3 are $+1$, and 3 are -1 .

We must note that any cyclic transposition of the numbers 1, 2, 3 can be obtained from 1, 2, 3 by an even number of transpositions of two subscripts, and any cyclic transposition of the numbers 3, 2, 1 can be obtained from 1, 2, 3 by an odd number of transpositions of two subscripts. Indeed, by changing, for instance, the places of 1 and 2 in the sequence 1, 2, 3 (which yields 2, 1, 3), i.e. a cyclic transposition of the numbers 3, 2, 1), and then changing the places of 1 and 3, we obtain the transposition 2, 3, 1.

Consequently, the values of the symbol ε_{ikl} can be determined as follows: (1) they are zero if the values of at least two subscripts coincide, (2) they are $+1$ or -1 depending on whether the sequence i, k, l can be obtained from the sequence 1, 2, 3 by an even or an odd number of transpositions.

We can also use the following rule to determine the sign of ε_{ikl} . When a larger number is ahead of a smaller one in a transposition, we shall call this a disorder. For example, in the transposition 2, 1, 3 there is one disorder—2 is ahead of 1, and in the transposition 3, 2, 1 there are three disorders—3 is ahead of 1, 3 is ahead of 2, and 2 is ahead of 1. Let us assign the value of $+1$ to ε_{ikl} if the number of disorders in the transposition i, k, l is even, and -1 if it is odd. It is easy to see that all three rules for determining the sign of ε_{ikl} which we have considered give the same result.

Let us prove the following very useful relation between the symbols ε and δ :

$$\sum_l \varepsilon_{ikl} \varepsilon_{mnl} = \delta_{im} \delta_{kn} - \delta_{in} \delta_{km} \quad (\text{VI.16})$$

We shall expand the sum on the left-hand side:

$$\varepsilon_{ik1} \varepsilon_{mn1} + \varepsilon_{ik2} \varepsilon_{mn2} + \varepsilon_{ik3} \varepsilon_{mn3} \quad (\text{VI.17})$$

and determine the values of the subscripts i, k, m , and n at which this sum is non-zero. It is evident that for at least one term to differ from zero, the conditions

$$i \neq k \text{ and } m \neq n \quad (\text{VI.18})$$

must be satisfied simultaneously. In addition, it is essential that

$$i = m, k = n \text{ or } i = n, k = m \quad (\text{VI.19})$$

Indeed, if the observance of the condition (VI.18) will not be attended by observance of the condition (VI.19), all three numbers 1, 2 and 3 will be present among the values of the first two subscripts of both factors in each of the terms in (VI.17). Therefore, the dummy index l in each of the terms will coincide with the value of one of the subscripts i, k, m and n , so that all the terms will vanish.

Let us combine the conditions (VI.18) and (VI.19) into one expressed by the formulas

$$i = m \neq k = n \quad (\text{VI.20})$$

$$i = n \neq k = m \quad (\text{VI.21})$$

In the case corresponding to relations (VI.20), the sum (VI.17) becomes

$$\varepsilon_{mn1}\varepsilon_{nm1} + \varepsilon_{mn2}\varepsilon_{mn2} + \varepsilon_{mn3}\varepsilon_{mn3}$$

It is obvious that only one term (for which m , n , and l are different) will be non-zero, and it equals $+1$.

In the case corresponding to relations (VI.21), the sum (VI.17) becomes

$$\varepsilon_{nm1}\varepsilon_{mn1} + \varepsilon_{nm2}\varepsilon_{mn2} + \varepsilon_{nm3}\varepsilon_{mn3}$$

In this sum too, only one term is non-zero, and it equals the product of $+1$ and -1 , i.e. -1 (upon the transposition of two subscripts ε_{nml} changes its sign).

Now let us turn to the right-hand side of formula (VI.16), i.e. to the expression

$$\delta_{im}\delta_{kn} - \delta_{in}\delta_{km} \quad (\text{VI.22})$$

If $i = k$ (or $m = n$), this expression becomes $\delta_{km}\delta_{kn} - \delta_{kn}\delta_{km}$ (or $\delta_{in}\delta_{kn} - \delta_{in}\delta_{kn}$). Both these expressions are zero. It thus follows that for expression (VI.22) to be other than zero, the following conditions must be observed simultaneously:

$$i \neq k \text{ and } m \neq n \quad (\text{VI.23})$$

[compare with (VI.18)]. In addition, one of the following two conditions must be observed:

$$i = m, \quad k = n \quad (\text{VI.24})$$

$$i = n, \quad k = m \quad (\text{VI.25})$$

When the condition (VI.24) is satisfied, the first term of expression (VI.22) is $+1$. Since $i = m$ and $m \neq n$ [see (VI.23)], we have $i \neq n$, and the second term of expression (VI.22) vanishes. Consequently, when the condition (VI.24) is observed, expression (VI.22) is $+1$. The combination of the conditions (VI.23) and (VI.24) is equivalent to the condition (VI.20) in which, as we have established, the left-hand side of formula (VI.16) also becomes equal to $+1$.

When the condition (VI.25) is observed, which in combination with (VI.23) is equivalent to the condition (VI.21), expression (VI.22) becomes equal to -1 .

We have thus proved relation (VI.16).

Using the symbol ε_{ikh} , we can write the combination of relations (VI.13) as a single expression:

$$[\mathbf{e}_i \mathbf{e}_k] = \sum_l \varepsilon_{ikh} \mathbf{e}_l \quad (\text{VI.26})$$

Indeed, when $i = 1$ and $k = 2$, only the addend $\varepsilon_{123} \mathbf{e}_3$ equal to \mathbf{e}_3 will be non-zero, when $i = 2$ and $k = 3$, only the addend $\varepsilon_{231} \mathbf{e}_1 = \mathbf{e}_1$ will be non-zero, and, finally, when $i = 3$ and $k = 1$, the addend $\varepsilon_{312} \mathbf{e}_2 = \mathbf{e}_2$ will be non-zero.

The expression (VI.26) gives even more than a combination of three relations (VI.13). It contains nine relations. It follows from it that the vector product of any unit vector by itself is zero—when $i = k$, all the addends, on the right-hand side of formula (VI.26) vanish. In addition, (VI.26) contains expressions obtained from (VI.13) by transposition of the multipliers. For instance, when $i = 2$ and $k = 1$, the addend $\varepsilon_{213} \mathbf{e}_3 = -\mathbf{e}_3$ on the right in (VI.26) is non-zero, etc.

Let us form the scalar product of the vectors $\mathbf{a} = \sum \mathbf{e}_i a_i$ and $\mathbf{b} = \sum \mathbf{e}_k b_k$:

$$ab = \left(\sum_i \mathbf{e}_i a_i \right) \left(\sum_k \mathbf{e}_k b_k \right)$$

Using the property of distributivity, we can write

$$ab = \sum_{i,k} \mathbf{e}_i \mathbf{e}_k a_i b_k = \sum_{i,k} \delta_{ik} a_i b_k$$

[see formula (VI.11)]. In accordance with the definition of δ_{ik} , in the last sum only addends having identical values of the subscripts i and k are non-zero. Therefore

$$ab = \sum_i a_i b_i \quad (\text{VI.27})$$

or, going over to conventional symbols, we have

$$ab = a_x b_x + a_y b_y + a_z b_z \quad (\text{VI.28})$$

For the given vectors \mathbf{a} and \mathbf{b} , their projections onto the coordinate axes depend on the choice of the coordinate system, but the product ab itself does not depend on this choice. We thus conclude that the expression $a_x b_x + a_y b_y + a_z b_z$ is an invariant, i.e. a quantity identical in all coordinate systems.

Assume that we have been given a set of three numbers u, v, w about which we know that in combination with the projections of a vector \mathbf{a} they yield a scalar, i.e. an invariant:

$$ua_x + va_y + wa_z = \text{inv}$$

On the basis of what has been said above, we can assert that u , v , and w are the components¹ of a vector.

Let us form the vector product of the vectors $\mathbf{a} = \sum \mathbf{e}_i a_i$ and $\mathbf{b} = \sum \mathbf{e}_k b_k$:

$$[\mathbf{ab}] = [(\sum_i \mathbf{e}_i a_i), (\sum_k \mathbf{e}_k b_k)]$$

Owing to distributivity, we can write

$$[\mathbf{ab}] = \sum_{i, k} [\mathbf{e}_i \mathbf{e}_k] a_i b_k$$

Let us replace $[\mathbf{e}_i \mathbf{e}_k]$ in accordance with formula (VI.26):

$$[\mathbf{ab}] = \sum_{i, k} a_i b_k \sum_l \varepsilon_{ikh} \mathbf{e}_l = \sum_{i, k, l} \varepsilon_{ikh} a_i b_k \mathbf{e}_l$$

The vector product can thus be written as

$$[\mathbf{ab}] = \sum_{i, k, l} \varepsilon_{ikh} a_i b_k \mathbf{e}_l \quad (\text{VI.29})$$

Of the 27 addends of this sum, only six are non-zero. Writing them out, we obtain

$$[\mathbf{ab}] = a_1 b_2 \mathbf{e}_3 + a_2 b_3 \mathbf{e}_1 + a_3 b_1 \mathbf{e}_2 - a_3 b_2 \mathbf{e}_1 - a_2 b_1 \mathbf{e}_3 - a_1 b_3 \mathbf{e}_2$$

Finally, combining terms with identical unit vectors, we arrive at the expression

$$[\mathbf{ab}] = \mathbf{e}_1 (a_2 b_3 - a_3 b_2) + \mathbf{e}_2 (a_3 b_1 - a_1 b_3) + \mathbf{e}_3 (a_1 b_2 - a_2 b_1) \quad (\text{VI.30})$$

that can be written in the form of a determinant (see Appendix VIII)

$$[\mathbf{ab}] = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (\text{VI.31})$$

or in conventional notation

$$[\mathbf{ab}] = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad (\text{VI.32})$$

We must note that according to (VI.29), the l -th component of a vector product is determined by the formula

$$[\mathbf{ab}]_l = \sum_{i, k} \varepsilon_{ikh} a_i b_k = \sum_{i, k} \varepsilon_{lik} a_i b_k$$

(we have performed a cyclic transposition of the subscripts on ε , which, as is known, does not change the numerical value of this

¹ For brevity's sake, we shall use this term for the projections of a vector onto the coordinate axes.

symbol). To deal with the more customary sequence of letter subscripts, we shall write an expression for the i -th component of a vector product:

$$[ab]_i = \sum_{k,l} \varepsilon_{ikl} a_k b_l \quad (\text{VI.33})$$

Let us prove formula (VI.5) using relation (VI.16). For this purpose, we shall write the vector product in accordance with formula (VI.29):

$$\mathbf{d} = [\mathbf{a}, [\mathbf{bc}]] = \sum_{k,l,i} \varepsilon_{kli} a_k [\mathbf{bc}]_{pr,l} \mathbf{e}_i$$

Now, we shall substitute for $[\mathbf{bc}]_{pr,l}$ its expression obtained from formula (VI.33):

$$\mathbf{d} = \sum_{k,l,i} \varepsilon_{kli} a_k \mathbf{e}_i \sum_{m,n} \varepsilon_{lmn} b_m c_n$$

Let us perform a cyclic transposition of the subscripts on the symbols ε so that their common subscript l will be in the last place. In addition, let us group the factors so that summation over l is performed first of all:

$$\mathbf{d} = \sum_{i,k,m,n} \mathbf{e}_i a_k b_m c_n \sum_l \varepsilon_{ikl} \varepsilon_{mnl} = \sum_{i,k,m,n} \mathbf{e}_i a_k b_m c_n (\delta_{im} \delta_{kn} - \delta_{in} \delta_{km})$$

[We have employed relation (VI.16)]. Further transformations yield

$$\begin{aligned} \mathbf{d} &= \sum_{i,k} \mathbf{e}_i a_k \sum_m \delta_{im} b_m \sum_n \delta_{kn} c_n - \sum_{i,k} \mathbf{e}_i a_k \sum_n \delta_{in} c_n \sum_m \delta_{km} b_m \\ &= \sum_{i,k} \mathbf{e}_i a_k b_i c_k - \sum_{i,k} \mathbf{e}_i a_k c_i b_k \\ &= \sum_i \mathbf{e}_i b_i \sum_k a_k c_k - \sum_i \mathbf{e}_i c_i \sum_k a_k b_k = \mathbf{b}(\mathbf{ac}) - \mathbf{c}(\mathbf{ab}) \end{aligned}$$

Q.E.D.

3. True Vectors and Pseudovectors. Two kinds of vectors are distinguished: polar (or true) and axial vectors, also known as **pseudovectors**¹. Upon inversion of the coordinate axes, i.e. when the directions of the coordinate axes are reversed (Fig. VI.3), the components of a true vector change their sign. This signifies that such a vector upon inversion remains unchanged. The components of a pseudovector upon inversion do not change their sign. This signifies that a pseudovector upon inversion reverses its direction (i.e. changes its sign).

Inspection of Fig. VI.3 shows that upon inversion of the coordinate axes, a right-hand system of coordinates transforms into a left-hand one. The distinction between a true vector and a pseudovector can therefore be defined as follows: a true (polar) vector does not

¹ "Pseudo" is a prefix meaning false or sham.

change upon a transition from a right-hand system of coordinates to a left-hand one, whereas a pseudovector reverses its direction in such a transition. It will be shown in Appendix X that a pseudovector is an antisymmetric second-rank tensor.

If both vectors \mathbf{a} and \mathbf{b} are true, the components of the vector product (VI.30) upon inversion do not change their sign (a_i and b_h separately change their sign, but their product remains unchanged). Consequently, the vector product of true vectors is a pseudovector.

Scalars must also be divided into two kinds: **true scalars** and **pseudoscalars**. True scalars do not change in a transition from a right-hand coordinate system to a left-hand one (or upon inversion of the coordinate axes). They include mass, electric charge, and temperature. Pseudoscalars change their sign in a transition from a right-hand coordinate system to a left-hand one. They include the scalar expressions obtained as a result of mathematical operations on vectors. For example, the scalar product [see (VI.27)] of a true vector and a pseudovector changes its sign upon inversion and, consequently, is not a true scalar, but a pseudoscalar.

If the vectors \mathbf{a} , \mathbf{b} , \mathbf{c} are true, expression (VI.3) will be a pseudoscalar—it changes its sign upon inversion. Hence, the scalar triple product of true vectors is a pseudoscalar.

4. Transformations of Vector Components. Let us find formulas for the transformation of the components of a vector in a transition from one coordinate system to another. Let us take two systems of Cartesian coordinates K and K' , setting them by their unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. The arbitrary vector \mathbf{a} can be written as $\mathbf{a} = \sum \mathbf{e}_h a_h$, where a_h are the projections of \mathbf{a} onto the axes of the system K , or as $\mathbf{a} = \sum \mathbf{e}'_h a'_h$, where a'_h are the projections of \mathbf{a} onto the axes of the system K' . Hence,

$$\sum \mathbf{e}'_h a'_h = \sum \mathbf{e}_h a_h \quad (\text{VI.34})$$

We multiply (VI.34) by the unit vector \mathbf{e}'_i :

$$\sum \mathbf{e}'_i \mathbf{e}'_h a'_h = \sum \mathbf{e}'_i \mathbf{e}_h a_h \quad (\text{VI.35})$$

By (VI.11), we have $\mathbf{e}'_i \mathbf{e}'_h = \delta_{ih}$. Consequently, of the three addends on the left-hand side, only the one with $h = i$ that is equal to $\delta_{ii} a'_i = a'_i$ will be non-zero.

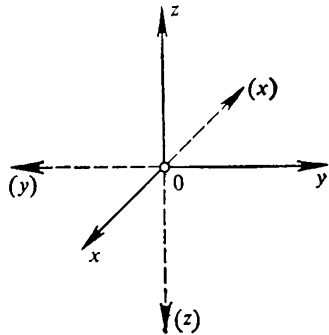


Fig. VI.3.

The scalar product $\mathbf{e}_i' \mathbf{e}_k$ equals the cosine of the angle between the axis x_i' of the system K' and the axis x_k of the system K . Designating this cosine by the symbol α_{ik} , we can write

$$\alpha_{ik} = \mathbf{e}_i' \mathbf{e}_k = \cos(x_i', x_k) \quad (i, k = 1, 2, 3) \quad (\text{VI.36})$$

Using this notation, we can write relation (VI.35) as

$$a_i' = \sum_k \alpha_{ik} a_k \quad (i = 1, 2, 3) \quad (\text{VI.37})$$

Formula (VI.37) allows us to calculate the projections of the vector \mathbf{a} onto the axes of the system K' according to the known projections of \mathbf{a} onto the axes of the system K . To obtain formulas for the reverse transformation (from K' to K), let us multiply (VI.34) by the unit vector \mathbf{e}_i . Repeating the reasoning that led us to formula (VI.37), we obtain

$$a_i = \sum_k \alpha_{ki} a_k' \quad (i = 1, 2, 3) \quad (\text{VI.38})$$

Formulas (VI.37) and (VI.38) differ only in that in one case summation is performed over the second subscript of α_{ik} , and in the other case over the first one.

The nine quantities α_{ik} are not independent. Let us form the sum $\sum_m \alpha_{im} \alpha_{km}$. Taking (VI.36) into account, we obtain

$$\sum_m \alpha_{im} \alpha_{km} = \sum_m (\mathbf{e}_i' \mathbf{e}_m) (\mathbf{e}_k' \mathbf{e}_m)$$

The quantity $\mathbf{e}_i' \mathbf{e}_m$ can be considered as the projection of the vector \mathbf{e}_i' onto the axis x_m of the system K ; similarly, $\mathbf{e}_k' \mathbf{e}_m$ is the projection of the vector \mathbf{e}_k' onto the axis x_m . Hence, the sum on the right can be written as

$$\sum_m (\mathbf{e}_i')_{pr, x_m} (\mathbf{e}_k')_{pr, x_m} = \mathbf{e}_i' \mathbf{e}_k = \delta_{ik}$$

[see formula (VI.27)]. Consequently,

$$\sum_m \alpha_{im} \alpha_{km} = \delta_{ik} \quad (\text{VI.39})$$

It can be proved in a similar way (we invite our reader to do this) that

$$\sum_m \alpha_{mi} \alpha_{mh} = \delta_{ih} \quad (\text{VI.40})$$

Transformations (VI.37) and (VI.38) can be adopted as a definition of a vector: *a vector is defined to be a set of quantities a_1, a_2, a_3 , which upon a transition from one coordinate system to another are transformed by formulas (VI.37) and (VI.38) where α_{ik} are quantities determined by formula (VI.36).*

The last definition of a vector can readily be extended to a space with any number of dimensions. Assume that we have an n -dimensional space. Let us take a system of coordinates whose axes are mutually perpendicular in this space. This signifies that the unit vectors e_1, e_2, \dots, e_n of the axes satisfy the condition (VI.11). A vector in n -dimensional space (an n -vector) is thus defined to be a set of n quantities a_1, a_2, \dots, a_n that upon a transition from one coordinate system to another are transformed by formulas (VI.37) and (VI.38). In summation, the dummy index runs through n values instead of 3. The number of equations in (VI.37) and (VI.38) will also be n instead of 3.

A scalar product of vectors can also be readily generalized for an n -dimensional space. Similar to (VI.27), we shall call the expression

$$ab = \sum_{i=1}^n a_i b_i \quad (\text{VI.41})$$

which is an invariant, the scalar product of two vectors having the components a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n . Vectors whose scalar product is zero are said to be mutually orthogonal (or mutually perpendicular).

The concept of a vector product cannot be extended to spaces with other than three dimensions.

Inversion of the coordinate axes (see Fig. VI.3) can be treated as a transformation from the system K to the system K' whose coefficients have the values

$$\alpha_{ik} = \begin{cases} -1 & \text{when } i = k \\ 0 & \text{when } i \neq k \end{cases}$$

or

$$\alpha_{ik} = -\delta_{ik} \quad (\text{VI.42})$$

By formula (VI.37), the components of a vector upon inversion are transformed according to the law

$$a'_i = -\sum_k \delta_{ik} a_k = -a_i$$

i.e. reverse their sign (this was already mentioned on an earlier page).

Let us find the law of transformation of the components of a vector product upon inversion of the coordinate axes. We write expression (VI.33) for the system of coordinates K' (obtained as a result of inversion of the axes of the system K):

$$[ab]'_i = \sum_{k,l} \varepsilon'_{ikl} a'_k b'_l = \sum_{k,l} \varepsilon_{ikl} a_k b_l \quad (\text{VI.43})$$

We have taken advantage of the circumstance that the quantities ε_{ikh} are determined in the same way for all coordinate systems, owing to which upon any transformations of the coordinates, we have

$$\varepsilon'_{ikh} = \varepsilon_{ikh} \quad (\text{VI.44})$$

Let us express a'_h and b'_l in formula (VI.43) in terms of the unprimed components of the relevant vectors using relation (VI.37)

$$[\mathbf{ab}]'_i = \sum_{h,l} \varepsilon_{ikh} \sum_m (-\delta_{hm}) a_m \sum_p (-\delta_{lp}) b_p = \sum_{h,l,m,p} \varepsilon_{ikh} \delta_{hm} \delta_{lp} a_m b_p$$

Summation over the subscripts m and p yields

$$[\mathbf{ab}]'_i = \sum_{h,l} \varepsilon_{ikh} a_h b_l$$

In accordance with (VI.33), the last expression is $[\mathbf{ab}]_i$. We have thus established that

$$[\mathbf{ab}]'_i = [\mathbf{ab}]_i$$

i.e. that the components of a vector product do not change in inversion. A vector product of true vectors is therefore a pseudovector.

Let us write a scalar triple product of three vectors. By formulas (VI.27) and (VI.33), we have

$$\mathbf{a}[\mathbf{bc}] = \sum_i a_i [\mathbf{bc}]_i = \sum_i a_i \sum_{h,l} \varepsilon_{ihl} b_h c_l = \sum_{i,h,l} \varepsilon_{ihl} a_i b_h c_l \quad (\text{VI.45})$$

Let us see how this quantity behaves in inversion. In the system K' , we have

$$\begin{aligned} (\mathbf{a}[\mathbf{bc}])' &= \sum_{i,h,l} \varepsilon_{ihl} a'_i b'_h c'_l \\ &= \sum_{i,h,l} \varepsilon_{ihl} \sum_m (-\delta_{im}) a_m \sum_p (-\delta_{hp}) b_p \sum_s (-\delta_{ls}) c_s \\ &= - \sum_{i,h,l} \varepsilon_{ihl} a_i b_h c_l = -(\mathbf{a}[\mathbf{bc}]) \end{aligned}$$

We have obtained a result we already know: the scalar triple product of true vectors upon inversion changes its sign, i.e. is a pseudoscalar.

5. Increment of a Vector in Rotation. Let us find the increment which the vector \mathbf{a} obtains in rotation through an infinitely small angle $d\varphi$. We introduce two coordinate systems K and K' which we choose so that their axes z and z' coincide with the vector $d\varphi$ (Fig. VI.4). Assume that the system K' turns together with the vector \mathbf{a} through the angle $d\varphi$ relative to the system K . Relative to the system K' , the vector \mathbf{a} remains constant, while relative to the system K it receives the increment $d\mathbf{a}$.

We shall first assume that the tail of the vector \mathbf{a} is on the z -axis (Fig. VI.4). If \mathbf{a} is initially in the plane yz , the increment $d\mathbf{a}$ is collin-

ear to the x -axis. The magnitude of this increment, as can be seen from the figure, is $a \sin \alpha d\varphi$. It follows from the above that the increment of the vector a can be written as

$$da = [d\varphi, a] \quad (\text{VI.46})$$

We shall prove that the formula we have found also holds with an arbitrary arrangement of the vector a relative to the coordinate systems K and K' . Let us introduce the unit vectors e_x, e_y, e_z of the system K and the unit vectors e'_x, e'_y, e'_z of the system K' . The vector a can therefore be set by the expression

$$a = e_x a_x + e_y a_y + e_z a_z \quad (\text{VI.47})$$

or the expression

$$a = e'_x a'_x + e'_y a'_y + e'_z a'_z \quad (\text{VI.48})$$

where a_x, a_y, a_z are the projections of the vector a onto the axes of the system K , and a'_x, a'_y, a'_z are the projections of a onto the axes of the system K' .

When the vector turns together with the system K' through the angle $d\varphi$, it receives an increment relative to K , that can be written as the increment of expression (VI.47):

$$da = e_x da_x + e_y da_y + e_z da_z$$

or as an increment of expression (VI.48):

$$da = a'_x de'_x + a'_y de'_y + a'_z de'_z \quad (\text{VI.49})$$

where de'_x, de'_y, de'_z are the increments of the unit vectors of the system K' observed in the system K (remember that the projections a'_x, a'_y, a'_z remain unchanged upon rotation).

With the direction of the z' -axis we have chosen, the increment of the unit vector e'_z vanishes ($de'_z = 0$). Figure VI.5 shows the increments de'_x and de'_y contained by the unit vectors e'_x and e'_y when the coordinate system K' turns through the angle $d\varphi$. Examination of the figure shows that the direction of de'_x coincides with the direction of the unit vector e'_y . The magnitude of de'_x , on the other hand, equals $d\varphi$ (the magnitude, i.e. the length of any unit vector is unity). Consequently, the increment of the unit vector e'_x observed in the system K can be written as

$$de'_x = e'_y d\varphi$$

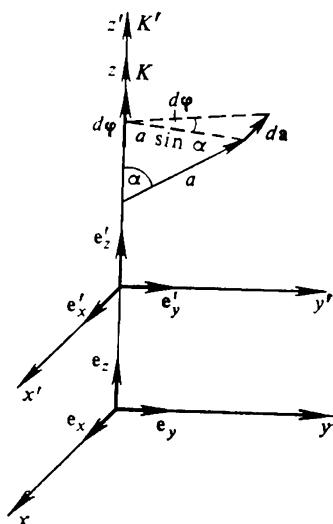


Fig. VI.4.

Similar reasoning results in the formula

$$de'_y = -e'_x d\varphi$$

the minus sign is due to the fact that the vectors de'_y and e'_x are directed oppositely).

Introducing the values of the unit vector increments we have found into formula (VI.49), we obtain

$$da = (a'_x e'_y - a'_y e'_x) d\varphi$$

We shall show that the expression we have found is equivalent to the vector product $[d\varphi, \mathbf{a}]$. To do this, we shall express the product in terms of the projections of the vectors being multiplied onto the axes of the frame K' , taking into account that $d\varphi$ is directed along the z' -axis. In accordance with formula (VI.32), we have

$$[d\varphi, \mathbf{a}] = \begin{vmatrix} e'_x & e'_y & e'_z \\ 0 & 0 & d\varphi \\ a'_x & a'_y & a'_z \end{vmatrix} = (a'_x e'_y - a'_y e'_x) d\varphi$$

We have thus arrived at formula (VI.46).

VII. Matrices

Definition. In Appendix VI, we obtained formulas for the transformation of vector components in a transition from the coordinate system K to the system K' :

$$a'_i = \sum_k \alpha_{ik} a_k \quad (i = 1, 2, 3) \quad (\text{VII.1})$$

$$a_i = \sum_k \alpha_{ki} a'_k \quad (i = 1, 2, 3) \quad (\text{VII.2})$$

[see formulas (VI.37) and (VI.38)].

The transition coefficients can be written in the form of a square table

$$A = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} \quad (\text{VII.3})$$

known as a **transformation matrix**. The quantities α_{ik} are the **matrix elements**. The first subscript indicates the number of the row in which the given element is, and the second subscript—the number of the column.

Let us come to an agreement about our notation. We shall denote the elements of a matrix by lower-case letters with two subscripts, and a matrix by the corresponding capital letter (for instance, the element α_{ih} of the matrix A ¹). We shall denote the components of a vector by lower-case italic letters with one subscript, and the vector itself by the same lower-case Roman (upright) letter in bold-face type (for instance, a_i is the component of a vector, and \mathbf{a} is the vector itself).

The operation (VII.1) of transforming the components of a vector can be written symbolically as the multiplication of a vector and a matrix:

$$\mathbf{a}' = A\mathbf{a} \quad (\text{VII.4})$$

The coefficients of the inverse transformation (VII.2) form the matrix

$$A^{-1} = \begin{vmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{vmatrix} \quad (\text{VII.5})$$

called an **inverse matrix**. Denoting the elements of an inverse matrix by the symbol α'_{ih} , we can write

$$\alpha'_{ih} = \alpha_{hi} \quad (\text{VII.6})$$

The matrix obtained from A by interchanging the rows with the columns is called a **transposed matrix** and is designated by \tilde{A} . If we denote the elements of a transposed matrix by the symbol $\tilde{\alpha}_{ih}$, we can write

$$\tilde{\alpha}_{ih} = \alpha_{hi} \quad (\text{VII.7})$$

A glance at formulas (VII.6) and (VII.7) shows that the matrix (VII.5) of the inverse transformation coincides with the transposed matrix (VII.3) of the direct transformation:

$$A^{-1} = \tilde{A} \quad (\text{VII.8})$$

Relation (VII.8) does not hold for any matrices². Matrices satisfying the condition (VII.8) are called **orthogonal**.

The inverse transformation (VII.2) is written symbolically as

$$\mathbf{a} = A^{-1}\mathbf{a}' \quad (\text{VII.9})$$

Without altering the formal mathematical aspect of the matter, relations (VII.4) and (VII.9) [in other words, relations (VII.1) and

¹ This is how the capital Greek letter "alpha" is written.

² In general, not any matrix has an inverse one. A matrix for which no inverse one exists is known as a singular or degenerate one. But even if a matrix is non-singular, its inverse and transposed matrices may not coincide.

(VII.2)] can be treated not as operations of transition from one coordinate system to another, but as operations transforming one vector into another, both vectors being considered in the same coordinate system. Having in view such an interpretation, we can write the formulas for transformation as follows:

$$\mathbf{b} = \mathbf{A}\mathbf{a} \quad (\text{VII.10})$$

$$\mathbf{a} = \mathbf{A}^{-1}\mathbf{b} \quad (\text{VII.11})$$

Hence, the matrix \mathbf{A} can be considered as a linear operator that by acting on the vector \mathbf{a} transforms it into the vector \mathbf{b} .

Let us write the transformations (VII.10) and (VII.11) in the explicit form, and for greater generality we shall consider that the vectors \mathbf{a} and \mathbf{b} are determined not in a three-dimensional space, but in a space with n dimensions. By analogy with (VII.1) and (VII.2), we obtain

$$b_i = \sum_{h=1}^n \alpha_{ih} a_h \quad (i = 1, 2, \dots, n) \quad (\text{VII.12})$$

$$a_i = \sum_{h=1}^n \alpha'_{ih} b_h \quad (i = 1, 2, \dots, n) \quad (\text{VII.13})$$

where α'_{ih} are the elements of the inverse transformation matrix (the matrix \mathbf{A}^{-1}). For an orthogonal matrix, $\alpha'_{ih} = \alpha_{hi}$.

The matrices \mathbf{A} and \mathbf{A}^{-1} will now have n rows and n columns, for example

$$\mathbf{A} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix} \quad (\text{VII.14})$$

The matrix (VII.14) is square—the number of rows in it equals the number of columns. In addition to square matrices, rectangular ones are also considered, in which the number of rows m does not equal the number of columns n :

$$\mathbf{A} = \mathbf{A}_{(m \ n)} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{vmatrix} \quad (\text{VII.15})$$

The first subscript on the matrix symbol indicates the number of rows, and the second—the number of columns. We shall drop these subscripts when this does not cause ambiguity.

Hence, in the general case, a matrix is defined to be a set of $m \cdot n$ elements arranged in the form of a rectangular array. The elements

of a matrix may be functions, numbers, or other quantities on which algebraic operations can be performed. A matrix with m rows and n columns is said to be an m by n matrix (written $m \times n$). An $m \times 1$ matrix, i.e. a matrix with one column, is called a **column matrix**, and a $1 \times n$ matrix, i.e. a matrix with one row, is called a **row matrix**.

Two matrices A and B are said to be equal ($A = B$) if the relevant elements of these matrices equal one another ($\alpha_{ik} = \beta_{ik}$).

The matrices A and B are considered to differ only in their sign ($A = -B$) if the relevant elements of these matrices are related by the expression $\alpha_{ik} = -\beta_{ik}$.

The square matrix (VII.14) (i.e. an $n \times n$ matrix) is a particular case of the matrix (VII.15). A matrix transforming a vector in an n -dimensional space into another vector in the same space will evidently be square.

If the elements of a square matrix satisfy the condition

$$\alpha_{ik} = \alpha_{ki} \quad (\text{VII.16})$$

the matrix is **symmetric**. A symmetric matrix obviously coincides with its transposed one:

$$A_{\text{sym}} = \tilde{A}_{\text{sym}} \quad (\text{VII.17})$$

A square matrix whose elements satisfy the condition

$$\alpha_{ik} = -\alpha_{ki} \quad (\text{VII.18})$$

is said to be **asymmetric** or **skew-symmetric**. An asymmetric matrix differs from its transposed one only in the sign:

$$A_{\text{asym}} = -\tilde{A}_{\text{asym}} \quad (\text{VII.19})$$

A square matrix in which only the elements α_{ik} with identical values of the subscripts i and k are non-zero is called **diagonal**. Such a matrix has the form

$$\Lambda = \left\| \begin{array}{cccc} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{array} \right\| \quad (\text{VII.20})$$

The elements of this matrix can be written as follows:

$$\lambda_{ik} = \lambda_k \delta_{ik} \quad (\text{VII.21})$$

where δ_{ik} is the Kronecker symbol [see (VI.12)].

If we change the coordinate system (i.e. the basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$), the components of the vectors \mathbf{a} and \mathbf{b} [see formula (VII.10)] become different. The elements of the matrix-operator will also change.

Sometimes (particularly, when the matrix A is symmetric), the basis can be chosen so that the matrix A becomes diagonal.

In a transition from one coordinate system to another, the elements of a matrix change, but the sum of the diagonal elements called the trace of the matrix (its symbol is $\text{Tr } A$) remains unchanged. Therefore, the trace of a matrix is identical in all coordinate systems, i.e. is an invariant:

$$\text{Tr } A = \sum_i \alpha_{ii} = \text{inv} \quad (\text{VII.22})$$

The determinant of the matrix [see (VIII.3)] also remains unchanged:

$$\det \|\alpha_{ik}\| = \text{inv} \quad (\text{VII.23})$$

Let us define a **unit** (or **identity**) matrix I which when multiplied by a vector according to the rule (VII.10) yields the same vector:

$$\mathbf{a} = I\mathbf{a}$$

It is a simple matter to see that the elements of a unit matrix must equal δ_{ik} [the substitution into (VII.12) of $\alpha_{ik} = \delta_{ik}$ leads to the relation $b_i = a_i$]. Hence,

$$I = \|\delta_{ik}\| = \left\| \begin{array}{cccc} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{array} \right\| \quad (\text{VII.24})$$

We must note that this matrix is diagonal.

Matrix Algebra. Matrices are algebraic objects lending themselves to addition, subtraction, and multiplication (the operation of matrix division does not exist).

The sum of two matrices A and B is the matrix $\Gamma = A + B$ whose elements are determined by the formula

$$\gamma_{ik} = \alpha_{ik} + \beta_{ik} \quad (\text{VII.25})$$

The difference of two matrices is the matrix $\Gamma = A - B$ with the elements

$$\gamma_{ik} = \alpha_{ik} - \beta_{ik} \quad (\text{VII.26})$$

It is evident that only matrices having the same number of rows and the same number of columns can be added and subtracted.

The product of the matrix A and the scalar η is defined to be the matrix $B = \eta A$ with the elements

$$\beta_{ik} = \eta \alpha_{ik} \quad (\text{VII.27})$$

Let us now consider the multiplication of matrices. Assume that the action of the matrix A on the vector \mathbf{a} transforms it into the vector \mathbf{b} , and the action of the matrix B on the vector \mathbf{b} transforms

it into the vector \mathbf{c} . It is natural to define the product of the matrices A and B as the matrix Γ which when acting on the vector \mathbf{a} transforms it into the vector \mathbf{c} . Hence,

$$\mathbf{b} = A\mathbf{a}, \text{ i.e. } b_m = \sum_k \alpha_{mk} a_k$$

$$\mathbf{c} = B\mathbf{b} = BA\mathbf{a}, \text{ i.e. } c_i = \sum_m \beta_{im} b_m = \sum_m \beta_{im} \sum_k \alpha_{mk} a_k = \sum_k a_k \sum_m \beta_{im} \alpha_{mk}$$

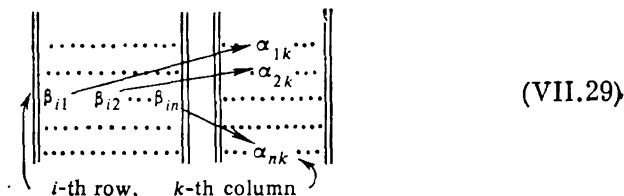
On the other hand,

$$\mathbf{c} = \Gamma\mathbf{a}, \text{ i.e. } c_i = \sum_k \gamma_{ik} a_k$$

A comparison of the two formulas for \mathbf{c} and c_i leads to the rule of matrix multiplication:

$$\Gamma = BA \text{ signifies that } \gamma_{ik} = \sum_m \beta_{im} \alpha_{mk} \quad (\text{VII.28})$$

According to this rule, to obtain an element of the matrix Γ at the intersection of the i -th row and the k -th column, we must multiply each element of the i -th row of the matrix B by the corresponding element of the k -th column of the matrix A and summate all the products. This can be explained by the following diagram:



We must note that matrix multiplication, in general, is not commutative, i.e.

$$BA \neq AB$$

Matrices for which the condition

$$BA = AB \quad (\text{VII.30})$$

is satisfied are called commutative.

It is easy to show that the product of matrices is associative:

$$(\Gamma B) A = \Gamma (BA) \quad (\text{VII.31})$$

This signifies that by first multiplying B and Γ , and then A and (ΓB) , we obtain the same result as we would by first multiplying the matrices A and B and then multiplying the matrix (BA) and Γ . In-

deed, according to the rule of matrix multiplication

$$\begin{aligned}\{(\Gamma B) A\}_{ik} &= \sum_m (\Gamma B)_{im} \alpha_{mk} = \sum_m \left(\sum_l \gamma_{il} \beta_{lm} \right) \alpha_{mk} \\ &= \sum_l \gamma_{il} \left(\sum_m \beta_{lm} \alpha_{mk} \right) = \sum_l \gamma_{il} (BA)_{lk} = \{\Gamma (BA)\}_{ik}\end{aligned}$$

(in the course of transformations we have changed the sequence of summation over the subscripts m and l). Consequently, the property (VII.31) has been proved.

Non-square (rectangular) matrices can also be multiplied by each other. Examination of the diagram (VII.29) shows that such matrices can be multiplied only when the number of columns of the matrix B (the second matrix¹) coincides with the number of rows of the matrix A (the first matrix). The product matrix will have the same number of rows as the second matrix (B) does, and the same number of columns as the first matrix (A) does. We shall explain this by the following example:

$$\begin{aligned}\begin{vmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \dots & \beta_{nn} \end{vmatrix} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} \end{vmatrix} \\ = \begin{vmatrix} (\sum \beta_{1k} \alpha_{k1}) & (\sum \beta_{1k} \alpha_{k2}) & (\sum \beta_{1k} \alpha_{k3}) \\ (\sum \beta_{2k} \alpha_{k1}) & (\sum \beta_{2k} \alpha_{k2}) & (\sum \beta_{2k} \alpha_{k3}) \\ \vdots & \vdots & \vdots \\ (\sum \beta_{nk} \alpha_{k1}) & (\sum \beta_{nk} \alpha_{k2}) & (\sum \beta_{nk} \alpha_{k3}) \end{vmatrix}\end{aligned}$$

If the second matrix is square, i.e. is an $n \times n$ matrix, and the first matrix has only one column with n elements, the product matrix also consists of one column with n elements:

$$\begin{vmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \dots & \beta_{nn} \end{vmatrix} \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{vmatrix} = \begin{vmatrix} (\sum \beta_{1k} \alpha_k) \\ (\sum \beta_{2k} \alpha_k) \\ \vdots \\ (\sum \beta_{nk} \alpha_k) \end{vmatrix} \quad (\text{VII.32})$$

When a column matrix is multiplied by a row matrix, the result is simply a number (or a function if the matrix elements are functions):

$$\begin{vmatrix} \beta_1 & \beta_2 & \dots & \beta_n \end{vmatrix} \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{vmatrix} = \sum_k \beta_k \alpha_k \quad (\text{VII.33})$$

¹ In the matrix product BA , the matrix A at the right should be considered as the first multiplier. It is multiplied by a vector first of all, and only then the second matrix B acts on the result.

Particularly, if we take the transposed matrix $\|\alpha\|$ as the matrix $\|\beta\|$, Eq. (VII.33) becomes

$$\|\alpha_1 \ \alpha_2 \ \dots \ \alpha_n\| \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{vmatrix} = \sum_k \alpha_k^2$$

Consequently, for the column matrix $A_{(n,1)}$, the following relation holds:

$$\tilde{A}_{(n,1)} A_{(n,1)} = \sum_{k=1}^n \alpha_k^2 \quad (\text{VII.34})$$

If we take the components of the vector a as the elements of a column matrix, and the matrix operator A as a square matrix, relation (VII.32) will become

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix} \begin{vmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{vmatrix} \quad (\text{VII.35})$$

where $b_i = \sum_k \alpha_{ik} a_k$ [compare with (VII.12)]. It is not at all difficult to see that relation (VII.35) is equivalent to relation (VII.10). Consequently, a vector can be represented as a column matrix.

Consider the product of the unit matrix I and an arbitrary matrix A . By rule (VII.28)

$$(AI)_{ik} = \sum_m \alpha_{im} \delta_{mk}$$

(δ_{mk} are the elements of the matrix I). In this sum, only one addend in which $m = k$ will be non-zero. Therefore, $(AI)_{ik} = \alpha_{ik}$. Similarly,

$$(IA)_{ik} = \sum_m \delta_{im} \alpha_{mk} = \alpha_{ik}$$

It follows from the above that multiplication by a unit matrix (with any sequence of the factors) does not change the matrix A :

$$IA = AI = A \quad (\text{VII.36})$$

Relation (VII.36) signifies that a unit matrix commutes with any matrix A .

It is obvious that by first applying to a vector the transformation A , and then its inverse transformation A^{-1} , we must return to the initial vector:

$$a = A^{-1} A a \quad (\text{VII.37})$$

This shows that the product of a direct and an inverse matrices must equal a unit matrix: $A^{-1}A = I^1$. The product of a direct and an inverse matrices is obviously commutative, hence

$$A^{-1}A = AA^{-1} = I \quad (\text{VII.38})$$

Having written the elements of the product of the matrices A and A^{-1} by formula (VII.28), we can find the relation between the elements of the direct and the inverse matrices:

$$\sum_m \alpha'_{im} \alpha_{mk} = \sum_m \alpha_{im} \alpha'_{mk} = \delta_{ik} \quad (\text{VII.39})$$

For an orthogonal matrix, i.e. one satisfying the condition (VII.8), we have $\alpha'_{ik} = \alpha_{ki}$ [see (VII.6)]. Making this substitution in (VII.39), we obtain

$$\sum_m \alpha_{mi} \alpha_{mk} = \delta_{ik} \quad (\text{VII.40})$$

$$\sum_m \alpha_{im} \alpha_{km} = \delta_{ik} \quad (\text{VII.41})$$

The elements of an orthogonal matrix thus satisfy relations (VII.39) and (VII.40) [compare with formulas (VI.39) and (VI.40)].

VIII. Determinants

Assume that we have the square matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \quad (\text{VIII.1})$$

Let us form the following expression from the elements of this matrix:

$$\varepsilon_{ik\dots m} a_{i1} a_{2k} \dots a_{nm} \quad (\text{VIII.2})$$

where i, k, \dots, m is a permutation of the numbers $1, 2, \dots, n$, and $\varepsilon_{ik\dots m}$ is a quantity equal to $+1$ if the number of disorders (inversions)² in the permutation i, k, \dots, m is even and to -1 if it is odd [compare with (VI.15)]. The number of permutations of n numbers taken n at a time is known to be $n!$ We can therefore compile $n!$ different expressions of the form given by (VII.2).

¹ This relation clarifies the symbol A^{-1} used for an inverse matrix (I is "unity").

² Recall that a disorder in a permutation is the fact that a larger number precedes a smaller one (see p. 320).

The sum of all the expressions having the form of (VIII.2) is denoted by the symbols

$$D_n = D(A) = \det ||a_{ik}|| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \\ = \sum_{(ik\dots m)} \varepsilon_{ik\dots m} a_{1i} a_{2k} \dots a_{nm} \quad (\text{VIII.3})$$

and is called the **determinant** corresponding to the matrix (VIII.1). The sum (VIII.3) is taken over all the permutations of the numbers i, k, \dots, m . Consequently, it contains $n!$ addends.

If we add to the definition of $\varepsilon_{ik\dots m}$ the condition that this quantity vanishes when the values of at least two of the n subscripts coincide, the determinant $D(A)$ can be evaluated as the sum

$$\sum_{i, k, \dots, m=1}^n \varepsilon_{ik\dots m} a_{1i} a_{2k} \dots a_{nm} \quad (\text{VIII.4})$$

in which all the subscripts i, k, \dots, m take on values from 1 to n .

A glance at (VIII.3) shows that a determinant can be written as an array similar to (VIII.1) with the difference that single vertical bars are used instead of double ones.

The number of rows (or columns) of a determinant is said to be its **order**.

We must note that the determinant of the diagonal matrix (VII.20) equals the product of its diagonal elements:

$$\det ||\lambda_k \delta_{ik}|| = \lambda_1 \lambda_2 \dots \lambda_n \quad (\text{VIII.5})$$

and the determinant of a unit matrix is unity:

$$D(I) = 1 \quad (\text{VIII.6})$$

Let us explain what has been said above using a third-order determinant as an example:

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

We can make $3! = 6$ permutations from the second subscripts of the elements of this determinant:

| | | |
|-----|-------------|---|
| 123 | 0 disorder | + |
| 231 | 2 disorders | + |
| 312 | 2 disorders | + |
| 321 | 3 disorders | — |
| 213 | 1 disorder | — |
| 132 | 1 disorder | — |

By formula (VIII.4), the expression

$$D_3 = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \\ - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

is a determinant of the third order.

If we delete the i -th row and the k -th column in an n -order determinant, we obtain a determinant of the $n - 1$ order known as the **minor** of the initial determinant corresponding to the element a_{ik} . This minor is customarily denoted by the symbol Δ_{ik} . The quantity

$$A_{ik} = (-1)^{i+k} \Delta_{ik} \quad (\text{VIII.7})$$

is called the **algebraic cofactor** of the element a_{ik} .

Properties of Determinants. We shall list the basic properties of determinants without proving them.

1. The value of a determinant does not change if corresponding rows and columns are interchanged:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{vmatrix}$$

or, more compactly,

$$\det \| a_{ik} \| = \det \| a_{ki} \| \quad (\text{VIII.8})$$

Interchanging of the rows and columns is called **transposition**. A determinant in which such a change has been made is said to be **transposed**. We can thus say that a transposed determinant equals the initial one.

Property 1 shows that the determinant of a transposed matrix equals the determinant of the initial matrix:

$$D(\tilde{A}) = D(A) \quad (\text{VIII.9})$$

Indeed, these determinants differ only in the rows and columns having been interchanged, which does not change the value of the determinant.

2. The sign of a determinant is changed if any two rows or any two columns are interchanged.

3. If two columns or rows are identical, the determinant is zero (this follows from the property 2).

4. A determinant is a linear form¹ of the elements of a row or a

¹ The linear form of the variables x_1, x_2, \dots, x_n is defined to be the linear homogeneous function of these variables, i.e. the expression

$$f = a_1x_1 + a_2x_2 + \dots + a_nx_n$$

column:

$$\det \|a_{ik}\| = \sum_{k=1}^n A_{ik} a_{ik} \text{ (linear form of the elements of the } i\text{-th row)} \quad (\text{VIII.10})$$

or

$$\det \|a_{ik}\| = \sum_{i=1}^n A_{ik} a_{ik} \text{ (linear form of the elements of the } k\text{-th column)} \quad (\text{VIII.11})$$

the algebraic cofactors (VIII.7) of the relevant elements being the coefficients A_{ik} of the linear forms (VIII.10) and (VIII.11).

5. If the elements of one row (or column) are multiplied by the algebraic cofactors of the elements of another row (or column) and the products obtained are summated, the sum will be zero (this sum is a determinant with two identical rows or columns; see the property 3).

Properties 4 and 5 can be combined in the form of the relations

$$\left. \begin{aligned} \sum_k A_{ik} a_{mk} &= \det \|a_{ik}\| \cdot \delta_{im} \\ \sum_i A_{ik} a_{im} &= \det \|a_{ik}\| \cdot \delta_{km} \end{aligned} \right\} \quad (\text{VIII.12})$$

Two more properties (6 and 7) follow directly from the property 4:

6. If all the elements of a row (or column) contain a common factor, it can be written before the determinant:

$$\begin{vmatrix} a_{11} & \dots & \beta a_{1k} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha a_{i1} & \dots & \alpha \beta a_{ik} & \dots & \alpha a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & \beta a_{nk} & \dots & a_{nn} \end{vmatrix} = \alpha \beta \begin{vmatrix} a_{11} & \dots & a_{1k} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & \dots & a_{ik} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nk} & \dots & a_{nn} \end{vmatrix} \quad (\text{VIII.13})$$

7. If the elements of a row (or column) are the sum of two (or more) addends, the determinant equals the sum of determinants in which the relevant addends are the elements of the given row (or column), for example,

$$\begin{vmatrix} a_{11} & \dots & a'_{1k} + a''_{1k} & \dots & a_{1n} \\ a_{21} & \dots & a'_{2k} + a''_{2k} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a'_{nk} + a''_{nk} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a'_{1k} & \dots & a_{1n} \\ a_{21} & \dots & a'_{2k} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a'_{nk} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & a''_{1k} & \dots & a_{1n} \\ a_{21} & \dots & a''_{2k} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a''_{nk} & \dots & a_{nn} \end{vmatrix} \quad (\text{VIII.14})$$

non-zero:

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0 \quad (\text{VIII.18})$$

We multiply the first of Eqs. (VIII.17) by A_{1m} —the cofactor of the element a_{1m} , the second equation by A_{2m} , \dots , the n -th equation by A_{nm} , and summate the expressions obtained. The result is

$$\sum_{i=1}^n A_{im} \sum_{k=1}^n a_{ik} x_k = \sum_{i=1}^n A_{im} b_i$$

We change the sequence of summation on the left-hand side of the equation:

$$\sum_{k=1}^n x_k \sum_{i=1}^n A_{im} a_{ik} = \sum_{i=1}^n A_{im} b_i$$

By the second of formulas (VIII.12), $\sum_{i=1}^n A_{im} a_{ik} = \det \| a_{ik} \| \delta_{mk} = D \delta_{mk}$, where D is the determinant of the system. Hence, the obtained relation can be written as

$$\sum_{k=1}^n x_k D \delta_{mk} = \sum_{i=1}^n A_{im} b_i$$

Summation on the left over k yields the product $x_m D$. A comparison of the sum on the right with formula (VIII.11) allows us to conclude that this sum is the determinant obtained from the determinant (VIII.18) by substituting the free terms of the system (VIII.17) for the elements of the m -th column. Denoting this determinant by the symbol $D^{(m)}$, we can write that

$$x_m D = D^{(m)}$$

whence

$$x_k = \frac{D^{(k)}}{D}$$

[m has been replaced with k for the subscript on x to be designated by the same letter in the given formula and in formula (VIII.17)].

We have arrived at Cramer's rule, which states: *if the determinant of a system is non-zero, it has one definite solution, the value of the unknown x_k being equal to a fraction whose denominator is the determinant D of the system, and whose numerator is the determinant $D^{(k)}$ obtained from D by replacing the elements of the k -th column with the*

free terms of the system:

$$x_k = \frac{\sum_{i=1}^n A_{ik} b_i}{D} = \frac{D^{(k)}}{D} = \left| \begin{array}{cccc} a_{11} & a_{12} & \dots & b_1 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & b_2 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & b_n & \dots & a_{nn} \end{array} \right| \quad (\text{VIII.19})$$

The solution of the system of equations (VIII.17) can be made very clear by using the following representation. The unknowns x_1, x_2, \dots, x_n can be considered as the components of a vector \mathbf{x} in an n -dimensional space, and the free terms b_1, b_2, \dots, b_n as the components of a given vector \mathbf{b} . Now the system (VIII.17) can be written symbolically as the relation

$$A\mathbf{x} = \mathbf{b} \quad (\text{VIII.20})$$

where A is a matrix compiled from the coefficients of Eqs. (VIII.17):

$$A = \left\| \begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{array} \right\|$$

Indeed, expanding relation (VIII.20) by formula (VII.12), we obtain n equations:

$$\sum_{k=1}^n a_{ik} x_k = b_i \quad (i = 1, 2, \dots, n)$$

coinciding with the system (VIII.17).

Consequently, the problem of finding the unknowns x_i can be formulated as follows: we are given the matrix A and the vector \mathbf{b} in an n -dimensional space; it is necessary to find a vector \mathbf{x} such that when multiplied by the matrix A transforms into the given vector \mathbf{b} .

We multiply Eq. (VIII.20) by the matrix A^{-1} . On the left, we obtain the required vector \mathbf{x} [see formula (VII.37)], and we arrive at the relation

$$\mathbf{x} = A^{-1}\mathbf{b} \quad (\text{VIII.21})$$

Hence, to find the solutions of the system of equations (VIII.17), we must proceed as follows: find the matrix that is the inverse of the system's matrix and substitute the elements of this matrix into the

of a matrix. If the number of rows m of a matrix differs from its number of columns n , the determinant (VIII.4) cannot be compiled for it. But by deleting certain rows and columns from the matrix, we can form a determinant from the remaining rows and columns. The determinants obtained in this way are said to be **included in the composition of the matrix**. The highest possible order of these determinants equals the minimum of the numbers m and n determining the size of the matrix, while the smallest order of these determinants is unity, the first-order determinants being elements of the matrix.

Assume that all the determinants of the order l in the composition of a matrix are zero. Hence, all the determinants of the order $(l + 1)$ are also zero (this follows from the property 4 of determinants). In other words, if all the determinants of the order l in the composition of a matrix are zero, all the higher-order determinants are also zero.

The highest order of a non-zero determinant in a matrix is called its **rank**. Hence, the fact that the rank of a matrix is r signifies that among the r -order determinants in the matrix at least one is non-zero; all the determinants of a higher order are zero here.

The concept of the rank, naturally, may also be applied to a square matrix. For example, with observance of condition (VIII.18), the rank of the matrix formed from the coefficients of the non-homogeneous system of equations (VIII.17) is n .

For the homogeneous system (VIII.26) to have a non-zero solution, its determinant must be zero. In other words, the rank of the matrix formed from the coefficients of the system must be less than n . Assume that the rank of the system's matrix is r (where $1 \leq r < n$). Here there are $n - r$ linearly independent solutions:

$$x_1^{(\alpha)}, x_2^{(\alpha)}, \dots, x_n^{(\alpha)} \quad (\alpha = 1, 2, \dots, n - r)$$

The values of the unknowns determined by the expressions

$$x_i = \sum_{\alpha} c_{\alpha} x_i^{(\alpha)} \quad (i = 1, 2, \dots, n) \quad (\text{VIII.27})$$

where c_{α} are arbitrary constants, will be the most general solution.

In the particular case when the rank of a system's matrix is $r = n - 1$, there is only one linearly independent solution. The following values of the unknowns can be proved to be this solution:

$$x_1 = cA_{k1}, \quad x_2 = cA_{k2}, \quad \dots, \quad x_n = cA_{kn} \quad (\text{VIII.28})$$

where A_{ki} is the algebraic cofactor of the element a_{ki} in the determinant D of the system, c is an arbitrary constant, and k is selected so that at least one of the A_{ki} 's (where $i = 1, 2, \dots, n$) is non-zero. The values of A_{ki} obtained with a different selection of k corresponding to the condition indicated above differ from one another by a common factor that can be included in the constant c . Hence, the form of the solution (VIII.28) does not depend on how k is selected.

Assume that a set of values $x_i = q_i$ satisfies the system (VIII.26). It is not difficult to see that the values $x_i = \lambda q_i$ (here λ is an arbitrary constant) also satisfy the system. This explains the presence of the factor c in formulas (VIII.28). It follows from the above that the system (VIII.26) uniquely determines only the ratios x_i/x_h , while the values of x_i themselves are determined to within an arbitrary factor.

The problem of solving a system of homogeneous equations can be given the following geometrical interpretation. We shall consider the set of quantities (x_1, x_2, \dots, x_n) as an n -vector \mathbf{x} ; similarly we shall consider the set of quantities $(a_{i1}, a_{i2}, \dots, a_{in})$ as an n -vector \mathbf{a}_i (there will be n such vectors). The system (VIII.26) can therefore be written as

$$\mathbf{a}_i \mathbf{x} = 0 \quad (i = 1, 2, \dots, n)$$

[see formula (VI.41)], and the problem itself formulated as follows: n vectors \mathbf{a}_i are set in an n -dimensional space. It is necessary to find a vector \mathbf{x} such that would be perpendicular to all the vectors \mathbf{a}_i .

It is evident that the multiplication of the vector \mathbf{x} by the scalar c does not violate its orthogonality to the vectors \mathbf{a}_i . Therefore, the unknowns x_1, x_2, \dots, x_n are determined by the system (VIII.26) to within the arbitrary factor c so that the value of one of the unknowns can be chosen arbitrarily (for instance, we can assume that $x_1 = 1$); now the values of the remaining unknowns will be determined uniquely (they will be expressed in terms of x_1).

IX. Quadratic Forms

A quadratic form f of the variables x_1, x_2, \dots, x_n is defined to be a homogeneous polynomial of the second degree in these variables. Such a polynomial can be written as

$$f = \sum_{i, h=1}^n a_{ih} x_i x_h \quad (\text{IX.1})$$

where

$$a_{ih} = a_{hi} \quad (\text{IX.2})$$

are constant quantities that may be either real or complex. If all the coefficients a_{ih} are real, the quadratic form is said to be real. If at least one of the coefficients is complex, the quadratic form is said to be complex.

A real quadratic form is called **positive definite** (**negative definite**) if this form has positive (negative) values for any real values of the variables x_1, x_2, \dots, x_n not equal to zero simultaneously.

In the following, we shall consider only real quadratic forms. The symmetric matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \quad (\text{IX.3})$$

compiled from the coefficients of the polynomial (IX.1) is called a matrix of a quadratic form. It is obvious that a quadratic form is completely determined by its matrix.

The determinant

$$D(A) = \det || a_{ik} || \quad (\text{IX.4})$$

compiled from the coefficients of a quadratic form is called its discriminant.

A quadratic form can be written as the product of three matrices:

$$f = || x_1 x_2 \dots x_n || \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{vmatrix} = \tilde{X} A X \quad (\text{IX.5})$$

where X is a column matrix, and \tilde{X} is its transposed matrix. Indeed, the product of a column matrix and a square matrix is a column matrix [see (VII.32)] whose elements in the case being considered are

$$\sum_k a_{ik} x_k \quad (i = 1, 2, \dots, n)$$

The product of a column matrix and a row matrix is simply a function [see (VII.33)]. In the given case, this function is

$$\sum_i x_i \sum_k a_{ik} x_k$$

The last expression can be seen to be identical to expression (IX.4).

A quadratic form such as

$$f_{\text{can}} = \sum_{k=1}^n \lambda_k x_k^2 \quad (\text{IX.6})$$

containing no terms with products of different variables is known as a canonical one. Such a form has a diagonal matrix and is therefore also called a **diagonal quadratic form**. It is obvious that if all the λ_k 's are greater (smaller) than zero, the form (IX.6) will be positive definite (negative definite).

Any quadratic form can be reduced to a diagonal form with the aid of a non-singular linear transformation¹. Let us go over from the

¹ A linear transformation B (where B is a matrix) is called non-singular when its determinant is non-zero: $D(B) \neq 0$.

variables x_1, x_2, \dots, x_n to the new variables y_1, y_2, \dots, y_n associated with the previous variables by means of the linear relations

$$x_i = \sum_k b_{ik} y_k \quad (\text{IX.7})$$

Let us consider the collection of quantities x_i and the collection of quantities y_k as column matrices. This allows us to write formula (IX.7) as

$$X = BY \quad (\text{IX.8})$$

where B is the matrix of the linear transformation whose elements are the coefficients b_{ik} .

Introducing the values of x_i determined by relations (IX.7) into formula (IX.1), we find an expression of the quadratic form in the new variables:

$$f = \sum_{i,k} a_{ik} \sum_l b_{il} y_l \sum_m b_{km} y_m = \sum_{l,m} y_l y_m \sum_{i,k} a_{ik} b_{il} b_{km} = \sum_{l,m} c_{lm} y_l y_m \quad (\text{IX.9})$$

where

$$c_{lm} = \sum_{i,k} a_{ik} b_{il} b_{km} \quad (\text{IX.10})$$

It is not difficult to see that the condition $a_{ik} = a_{ki}$ results in $c_{lm} = c_{ml}$.

Let us write expression (IX.10) as follows:

$$c_{lm} = \sum_i b_{il} \sum_k a_{ik} b_{km} = \sum_i \tilde{b}_{li} \sum_k a_{ik} b_{km}$$

[we have replaced the elements of the matrix B with the corresponding elements of the transposed matrix \tilde{B} , see formula (VII.7)]. According to (VII.28), $\sum_k a_{ik} b_{km}$ is $(AB)_{im}$ —an element of the matrix obtained by multiplying the matrices B and A . Similarly, $\sum_i \tilde{b}_{li} (AB)_{im}$ is $(\tilde{B}AB)_{lm}$ —an element of the matrix obtained by multiplying the matrices (AB) and \tilde{B} . Consequently, the matrix C whose elements are determined by formula (IX.10) can be written as

$$C = \tilde{B}AB \quad (\text{IX.11})$$

By (VIII.15), the determinant $D(C)$ of the matrix C is

$$D(C) = D(\tilde{B}) D(A) D(B)$$

Since the matrices \tilde{B} and B differ in the columns being interchanged with the rows, while the determinant does not change in this case [see (VIII.8)], we have $D(\tilde{B}) = D(B)$, and we can write

$$D(C) = D(A) [D(B)]^2 \quad (\text{IX.12})$$

Hence, in a linear transformation of variables, the discriminant of the quadratic form is multiplied by the square of the determinant of the transformation from the new variables to the initial ones.

If the transformation B is orthogonal [this signifies that the coefficients b_{ik} satisfy the conditions (VII.40) and (VII.41)], the transposed matrix coincides with the inverse one: $\tilde{B} = B^{-1}$ [see (VII.8)]. Consequently, for an orthogonal transformation, formula (IX.11) is as follows:

$$C = B^{-1}AB \quad (\text{IX.13})$$

The matrix C determined by relation (IX.13) is a matrix of a quadratic form in the new variables y_k . Let us find an orthogonal transformation B such that the matrix C will be diagonal, i.e. will have the form

$$C = \Lambda = \begin{vmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \lambda_n \end{vmatrix} \quad (\text{IX.14})$$

Now the quadratic form in the new variables y_k will be canonic [see formula (IX.6)].

We multiply both sides of formula (IX.13) by B . Since $BB^{-1} = I$, and multiplication by a unit matrix does not change the second factor [see (VII.36)], we arrive at the relation

$$BC = AB$$

or

$$\sum_m b_{im}c_{mk} = \sum_m a_{im}b_{mk} \quad (i, k = 1, 2, \dots, n) \quad (\text{IX.15})$$

In accordance with (IX.14), the elements c_{mk} can be written as $c_{mk} = \lambda_m \delta_{mk}$. Substitution of this value of c_{mk} into formula (IX.15) yields

$$\sum_m b_{im}\lambda_m\delta_{mk} = \sum_m a_{im}b_{mk} \quad (i, k = 1, 2, \dots, n)$$

In the sum on the left, only the addend with $m = k$ will be non-zero, and it equals $b_{ik}\lambda_k$. We thus arrive at the equation

$$b_{ik}\lambda_k = \sum_m a_{im}b_{mk} \quad (i, k = 1, 2, \dots, n) \quad (\text{IX.16})$$

Expression (IX.16) can be treated as a set of n^2 equations with n^2 unknowns b_{ik} . These equations can be divided into n groups (differing in the values of the subscript k). Each group consists of n equations differing in the values of the subscript i . Transferring all the terms in (IX.16) to one side, we can write the k -th group of equations

{see formulas (VII.36) and (VII.38)]. Consequently, the quadratic form (IX.19) in the new variables is

$$f = \sum_i y_i^2$$

Any orthogonal transformation of variables thus leaves a quadratic form of the kind given by (IX.19) unchanged.

Let us consider the real quadratic form

$$f = \sum_{k, m} a_{km} z_k^* z_m \quad (\text{IX.21})$$

where z_1, z_2, \dots, z_n are complex quantities¹. We shall represent z_m as $x_m + iy_m$, and z_k^* correspondingly as $x_k - iy_k$. Hence,

$$f = \sum_{k, m} a_{km} (x_k - iy_k) (x_m + iy_m) = \sum_{k, m} a_{km} (x_k x_m + y_k y_m) + i \sum_{k, m} a_{km} (x_k y_m - x_m y_k) = f_1 + i f_2 \quad (\text{IX.22})$$

Let us interchange the dummy indices k and m in the sum determining the imaginary part f_2 in (IX.22):

$$\begin{aligned} f_2 &= \sum_{k, m} a_{km} (x_k y_m - x_m y_k) = \sum_{m, k} a_{mk} (x_m y_k - x_k y_m) \\ &= - \sum_{k, m} a_{km} (x_k y_m - x_m y_k) = -f_2 \end{aligned}$$

{we have taken advantage of the fact that $a_{mk} = a_{km}$; see (IX.2)}. The relation $f_2 = -f_2$ is possible only when $f_2 = 0$. We have thus proved that the quadratic form (IX.21) has real values at any complex z 's. By (IX.22), let us write it as

$$f = \sum_{k, m} a_{km} x_k x_m + \sum_{k, m} a_{km} y_k y_m = f_0(x_k) + f_0(y_k) \quad (\text{IX.23})$$

where

$$f_0(\xi_k) = \sum_{k, m} a_{km} \xi_k \xi_m$$

(recall that a quadratic form is determined completely by its matrix and does not depend on the designation of the variables).

Simultaneous Reduction of Two Quadratic Forms to a Diagonal Form. Assume that we have two quadratic forms:

$$f_1 = \sum_{i, k} a_{ik} x_i x_k \quad (\text{IX.24})$$

$$f_2 = \sum_{i, k} b_{ik} x_i x_k \quad (\text{IX.25})$$

¹ Recall that a quadratic form with the real coefficients a_{km} is said to be real. The form (IX.21) is a particular case of the Hermitian form in which the coefficients, generally speaking, are complex and satisfy the condition $a_{ik} = a_{ki}^*$.

We shall show that if one of them, say f_1 , is positive definite, it is possible to find a linear transformation of the variables that reduces both forms to a diagonal one. We shall carry out the required transformation in several steps. First, using the orthogonal transformation F , we shall pass over to the variables v_i in which the form (IX.24) acquires a diagonal form:

$$f_1 = \sum_i \mu_i v_i^2$$

(such a transformation was considered earlier in detail).

Now let us pass over from the variables v_i to the variables

$$u_i = v_i \sqrt{\mu_i}$$

(it can readily be seen that with at least one $\mu_i \neq 1$, this transformation is not orthogonal). Since the form f_1 is positive definite, all the coefficients μ_i are positive so that the variables u_i will be real. In these variables

$$f_1 = \sum_i u_i^2$$

hence, I will be the matrix of f_1 .

Finally, let us go over with the aid of the orthogonal transformation G from the variables u_i to the variables y_i such that the form f_2 will become diagonal. The form f_1 here remains diagonal because, as was shown above, a quadratic form with the matrix I [see (IX.20)] does not change in any orthogonal transformation. Consequently, in the variables y_i , the quadratic forms (IX.24) and (IX.25) will be diagonal:

$$f_1 = \sum_i y_i^2, f_2 = \sum_i \lambda_i y_i^2 \quad (\text{IX.26})$$

The entire sequence of transformations can be represented by the diagram:

$$\begin{array}{ccccccc} \text{(transformation } F) & (\times \sqrt{\mu_i}) & \text{(transformation } G) & & & & \\ x_i & \rightarrow & v_i & \rightarrow & u_i & \rightarrow & y_i \\ \sum_{i,h} a_{ih} x_i x_h & \rightarrow & \sum_i \mu_i v_i^2 & \rightarrow & \sum_i u_i^2 & \rightarrow & \sum_i y_i^2 \\ \sum_{i,h} b_{ih} x_i x_h & \rightarrow & \sum_{i,h} b'_{ih} v_i v_h & \rightarrow & \sum_{i,h} b''_{ih} u_i u_h & \rightarrow & \sum_i \lambda_i y_i^2 \quad (\text{IX.27}) \end{array}$$

To establish a method of finding the coefficients λ_i , let us compile the auxiliary quadratic form

$$f = f_2 - \lambda f_1 = \sum_{i,h} (b_{ih} - \lambda a_{ih}) x_i x_h = \sum (\lambda_h - \lambda) y_h^2 \quad (\text{IX.28})$$

whose coefficients contain the parameter λ . The discriminant of this form in the variables x_i is

$$\det \| b_{ik} - \lambda a_{ik} \| \quad (\text{IX.29})$$

and in the variables y_i

$$\det \| (\lambda_k - \lambda) \delta_{ik} \| = (\lambda_1 - \lambda) (\lambda_2 - \lambda) \dots (\lambda_n - \lambda) \quad (\text{IX.30})$$

Denoting the transformation of the direct transition from the variables x_i to the variables y_i by the letter B (this transformation, generally speaking, will not be orthogonal), let us write the following expression for the matrix of the quadratic form (IX.28) in the variables y_i :

$$C = \tilde{B}AB$$

where A stands for the matrix of the form (IX.28) in the variables x_i [see formula (IX.11)]. According to (IX.12)

$$D(C) = D(A) [D(B)]^2$$

where $D(C)$ is the determinant (IX.30), $D(A)$ is the determinant (IX.29), and $D(B)$ is the non-zero determinant of the matrix of the transformation B in which the parameter λ is absent [$D(B) \neq 0$ because the transformation B is non-singular)].

Hence,

$$(\lambda_1 - \lambda) (\lambda_2 - \lambda) \dots (\lambda_n - \lambda) = \det \| b_{ik} - \lambda a_{ik} \| [D(B)]^2$$

The substitution of any of the values of λ_k for λ makes the left-hand side of the equation and, consequently, the factor $\det \| b_{ik} - \lambda a_{ik} \|$ vanish. Therefore, the quantities λ_k are the roots of the equation

$$\begin{vmatrix} b_{11} - \lambda a_{11} & b_{12} - \lambda a_{12} & \dots & b_{1n} - \lambda a_{1n} \\ b_{21} - \lambda a_{21} & b_{22} - \lambda a_{22} & \dots & b_{2n} - \lambda a_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} - \lambda a_{n1} & b_{n2} - \lambda a_{n2} & \dots & b_{nn} - \lambda a_{nn} \end{vmatrix} = 0 \quad (\text{IX.31})$$

The task of reducing the quadratic forms (IX.24) and (IX.25) to a diagonal form thus consists in finding the roots of Eq. (IX.31).

Let us consider in addition to the quadratic form in the variables x_i

$$f = \sum_{i,k} a_{ik} x_i x_k \quad (\text{IX.32})$$

the similar quadratic form in the variables \dot{x}_i

$$f' = \sum_{i,k} a_{ik} \dot{x}_i \dot{x}_k \quad (\text{IX.33})$$

where \dot{x}_i is the derivative of the variable x_i with respect to a parameter t . Let us call \dot{x}_i the i -th velocity.

In the linear transformation

$$x_i = \sum_m c_{im} y_m \quad (\text{IX.34})$$

from the variables x_i to the new variables y_i , the velocities experience the same transformation:

$$\dot{x}_i = \sum_m c_{im} \dot{y}_m \quad (\text{IX.35})$$

We express the forms (IX.32) and (IX.33) in the new variables. To do this, we introduce expressions (IX.34) and (IX.35) into formulas (IX.32) and (IX.33):

$$\begin{aligned} f &= \sum_{i,k} a_{ik} x_i x_k = \sum_{i,k} a_{ik} \sum_m c_{im} y_m \sum_l c_{kl} y_l \\ &= \sum_{m,l} y_m y_l \sum_{i,k} a_{ik} c_{im} c_{kl} = \sum_{m,l} b_{ml} y_m y_l \end{aligned}$$

where

$$b_{ml} = \sum_{i,k} a_{ik} c_{im} c_{kl} \quad (\text{IX.36})$$

Similarly

$$\begin{aligned} f' &= \sum_{i,k} a_{ik} \dot{x}_i \dot{x}_k = \sum_{i,k} a_{ik} \sum_m c_{im} \dot{y}_m \sum_l c_{kl} \dot{y}_l \\ &= \sum_{m,l} \dot{y}_m \dot{y}_l \sum_{i,k} a_{ik} c_{im} c_{kl} = \sum_{m,l} b_{ml} \dot{y}_m \dot{y}_l \end{aligned}$$

where b_{ml} has the same values (IX.36) as in the preceding case.

Hence, in any linear transformation from the variables x_i to the new variables y_i , the coefficients of the quadratic form of the velocities \dot{x}_i are transformed in exactly the same way as the coefficients of the similar quadratic form of x_i . On these grounds, the diagram (IX.27) can be modified as follows:

$$\begin{array}{ccccccc} \text{(transformation } F) & & (\times \sqrt{\mu_i}) & & \text{(transformation } G) & & \\ x_i & \rightarrow & v_i & \rightarrow & u_i & \rightarrow & y_i \\ \sum_{i,k} a_{ik} \dot{x}_i \dot{x}_k & \rightarrow & \sum_i \mu_i \dot{v}_i^2 & \rightarrow & \sum_i \dot{u}_i^2 & \rightarrow & \sum_i \dot{y}_i^2 \\ \sum_{i,k} b_{ik} x_i x_k & \rightarrow & \sum_{i,k} b'_{ik} v_i v_k & \rightarrow & \sum_{i,k} b''_{ik} u_i u_k & \rightarrow & \sum_i \lambda_i y_i^2 \end{array} \quad (\text{IX.37})$$

The quadratic form $\sum a_{ik} \dot{x}_i \dot{x}_k$ is assumed to be positive definite.

X. Tensors

1. Definition of a Tensor. To arrive at the concept of a tensor, let us consider the polarization of an anisotropic dielectric.

In an isotropic dielectric, the polarization \mathbf{P} is proportional to the electric field strength \mathbf{E} :

$$\mathbf{P} = \chi \mathbf{E} \quad (\text{X.1})$$

where χ is the dielectric susceptibility. According to (X.1), the vectors \mathbf{P} and \mathbf{E} are collinear.

In an anisotropic dielectric, the polarizability differs in different directions. As a result, the direction of the vector \mathbf{P} , generally

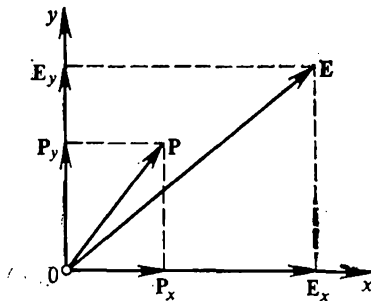


Fig. X.1.

speaking, does not coincide with that of the vector \mathbf{E} . Experiments show that in any anisotropic dielectric there are three mutually perpendicular directions such that when the direction of \mathbf{E} coincides with one of them, the vector \mathbf{P} is collinear with \mathbf{E} . These directions are called the principal ones. Let us direct the coordinate axes along the principal directions of a dielectric (Fig. X.1). The arbitrarily directed vector \mathbf{E} can be resolved

into the components E_x , E_y , and E_z (the last component is perpendicular to the plane of the drawing). The component E_x will set up the polarization $P_x = \chi_x E_x$ collinear with it, where χ_x is the susceptibility in the direction of the x -axis. Similarly, the other two components will set up $P_y = \chi_y E_y$ and $P_z = \chi_z E_z$. It is not difficult to note that with different values of χ_x , χ_y , and χ_z the resultant vector $\mathbf{P} = \mathbf{P}_x + \mathbf{P}_y + \mathbf{P}_z$ will not be collinear with \mathbf{E} .

Let us take an anisotropic dielectric which we shall consider to be a homogeneous unbounded medium. We associate with it a Cartesian coordinate system whose axes are oriented absolutely arbitrarily and coincide with none of the principal directions of the dielectric. With the field E_x directed along the x -axis, not only P_x , but also P_y and P_z will be non-zero, and

$$P_x = \chi_{xx} E_x, \quad P_y = \chi_{yx} E_x, \quad P_z = \chi_{zx} E_x \quad (\text{X.2})$$

where χ_{xx} , χ_{yx} , and χ_{zx} are coefficients of proportionality between E_x and the relevant components of \mathbf{P} .

Similarly, the fields E_y and E_z will cause the polarizations

$$\left. \begin{aligned} P_x &= \chi_{xy} E_y, & P_y &= \chi_{yy} E_y, & P_z &= \chi_{zy} E_y \\ P_x &= \chi_{xz} E_z, & P_y &= \chi_{yz} E_z, & P_z &= \chi_{zz} E_z \end{aligned} \right\} \quad (\text{X.3})$$

With a field \mathbf{E} not coinciding with any of the coordinate axes, E_x , E_y , and E_z will exist simultaneously so that all the P_i 's determined by formulas (X.2) and (X.3) will appear. Combining the

relevant components of the vector \mathbf{P} , we find that

$$\left. \begin{aligned} P_x &= \chi_{xx}E_x + \chi_{xy}E_y + \chi_{xz}E_z \\ P_y &= \chi_{yx}E_x + \chi_{yy}E_y + \chi_{yz}E_z \\ P_z &= \chi_{zx}E_x + \chi_{zy}E_y + \chi_{zz}E_z \end{aligned} \right\} \quad (\text{X.4})$$

Using numerical subscripts instead of letters, we can write equations (X.4) in a compact form:

$$P_i = \sum_k \chi_{ik} E_k \quad (i = 1, 2, 3) \quad (\text{X.5})$$

It follows from the above that to characterize an anisotropic dielectric, we must set nine quantities χ_{ik} (a single quantity χ was sufficient for an isotropic dielectric).

Let us now go over from the previous coordinate system x_1, x_2, x_3 (the system K) to a new system x'_1, x'_2, x'_3 (the system K') whose axes also do not coincide with the principal directions of the dielectric. We shall find out how the quantities χ_{ik} transform in such a transition. In the new system of coordinates, the equations relating P'_i and E'_k are similar to equations (X.5):

$$P'_i = \sum_k \chi'_{ik} E'_k \quad (\text{X.6})$$

Here χ'_{ik} are the nine quantities characterizing the dielectric in the new coordinate system.

By formulas (VI.37) and (VI.38), the components of the vector \mathbf{P} in the transition from the system K to the system K' are transformed by the formula

$$P'_i = \sum_l \alpha_{il} P_l \quad (\text{X.7})$$

and the components of the vector \mathbf{E} in the transition from the system K' to the system K are transformed by the formula

$$E_m = \sum_k \alpha_{km} E'_k \quad (\text{X.8})$$

(Recall that $\alpha_{ik} = \mathbf{e}'_i \mathbf{e}_k$ is the cosine of the angle between the i -th primed and the k -th unprimed coordinate axes.)

Let us replace P_l with E_m in (X.7) according to relation (X.5). The result is

$$P'_i = \sum_l \alpha_{il} P_l = \sum_l \alpha_{il} \sum_m \chi_{lm} E_m = \sum_{l,m} \alpha_{il} \chi_{lm} E_m$$

Now let us substitute into this equation E_m from formula (X.8):

$$P'_i = \sum_{l,m} \alpha_{il} \chi_{lm} \sum_k \alpha_{km} E'_k = \sum_k E'_k \sum_{l,m} \alpha_{il} \alpha_{km} \chi_{lm}$$

Comparing the expression obtained with expression (X.6), we find that

$$\chi'_{ik} = \sum_{l, m} \alpha_{il} \alpha_{km} \chi_{lm} \quad (\text{X.9})$$

The set of the nine quantities T_{ik} that transform in the transition from the coordinate system K to the system K' by the formula

$$T'_{ik} = \sum_{l, m} \alpha_{il} \alpha_{km} T_{lm} \quad (\text{X.10})$$

is known as a **tensor of the second rank** (or a tensor of the second order).

The reverse transformation (from the system K' to the system K) is performed by the formula

$$T_{ik} = \sum_{l, m} \alpha_{li} \alpha_{mk} T'_{lm} \quad (\text{X.11})$$

A tensor is written in one of the following three ways:

$$\mathbf{T} = (T_{ik}) = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \quad (\text{X.12})$$

The quantities T_{ik} are called the **components** of a tensor. The components T_{11} , T_{22} , and T_{33} are said to be **diagonal**.

Hence, the properties of an anisotropic dielectric are described by the dielectric susceptibility tensor

$$(\chi_{ik}) = \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix} \quad (\text{X.13})$$

Of special interest is the case when the coordinate axes coincide with the principal directions of a dielectric. Now the field component E_i sets up only the i -th component of the polarization, and

$$P_i = \chi_{ii} E_i \quad (i = 1, 2, 3)$$

A comparison with (X.5) leads us to the conclusion that only the diagonal tensor components are non-zero in the present case so that the dielectric susceptibility tensor is

$$(\chi_{ik}) = \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix} \quad (\text{X.14})$$

(we have left only one subscript on the non-zero tensor components because both subscripts are the same for diagonal components).

A tensor in which only the diagonal components are non-zero¹ is said to be **reduced to the principal axes**. The values of the diagonal components obtained in this case are called the **principal values of the tensor**.

We must note that in an isotropic dielectric all three principal values of the dielectric susceptibility tensor are the same: $\chi_1 = \chi_2 = \chi_3 = \chi$. For an isotropic dielectric, any three mutually perpendicular directions can be taken as its principal directions.

Tensors not only of the second rank, but also of other ranks are considered. For instance, a tensor of the third rank is a collection of 27 quantities T_{ihl} that transform in a transition from one coordinate system to another by the formula

$$T'_{ihl} = \sum_{m,p,s} \alpha_{im} \alpha_{hp} \alpha_{ls} T_{mps} \quad (\text{X.15})$$

Tensors of other ranks are determined similarly. A tensor of rank r has 3^r components. It is a simple matter to see that a vector is a tensor of the first rank (it has $3^1 = 3$ components), and a scalar is a tensor of the zeroth rank (it has $3^0 = 1$ component).

The concept of a tensor can readily be extended to an n -dimensional space. A tensor of rank r in such a space (an n -tensor of rank r) is defined to be a set of n^r quantities $T_{ih\dots p}$ (altogether r subscripts) that transform according to a formula differing from formula (X.15) only in that the dummy indices take on n values instead of three in summation.

We shall consider some more examples of tensors of rank two. Let us take two vectors **a** and **b** and form products of the kind

$$\Pi_{ik} = a_i b_k \quad (\text{X.16})$$

from their components. It is not difficult to see that these products are transformed by formula (X.10), i.e. have properties of the components of a tensor of the second rank.

The tensor

$$(\delta_{ik}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{X.17})$$

is called a **unit tensor**. By transformation formula (VI.39), its components in the new coordinate system are

$$\delta'_{ik} = \sum_{l,m} \alpha_{il} \alpha_{km} \delta_{lm} = \sum_l \alpha_{il} \alpha_{kl} = \delta_{ik}$$

[we have taken advantage of the property of the coefficients α_{ik} expressed by formula (X.39)]. The components of a unit tensor

¹ We must note that this is possible only with specially chosen coordinate axes.

are thus identical in all coordinate systems. Tensors having this property are said to be **invariant**.

2. Tensor Algebra. Let us consider the fundamental operations with tensors.

The sum of the tensors T_{ik} and G_{ik} is defined to be a tensor with the components

$$\Sigma_{ik} = T_{ik} + G_{ik} \quad (\text{X.18})$$

In accordance with (X.18), any tensor can be written as the sum of two (or more) tensors.

The product of the tensor T_{ik} and the scalar α is defined to be the tensor G_{ik} with the components

$$G_{ik} = \alpha T_{ik} \quad (\text{X.19})$$

The product of the tensors T_{ik} and G_{lm} is defined to be the tensor Π_{iklm} of the fourth rank with the components

$$\Pi_{iklm} = T_{ik} G_{lm} \quad (\text{X.20})$$

The product of tensors of other ranks is determined similarly. Particularly, we have considered [see (X.16)] the product of tensors of rank one, i.e. vectors. Examination of definition (X.20) shows that the rank of a product tensor equals the sum of the ranks of the multiplier tensors (tensors of different ranks can also be multiplied).

By the contraction of a tensor is meant the following operation: two subscripts on the tensor components are assumed to be the same and summation is performed over them. The quantity obtained as a result of such an operation is called the **contraction of the tensor**. Contraction obviously lowers the rank of a tensor by two units. The contraction operation can be used for tensors of a rank not lower than the second one¹. For a tensor of the second rank, its contraction is a tensor of the zero rank, i.e. a scalar. The latter is called the **trace of the tensor** (compare with the trace of a matrix, p. 334). It equals the sum of the diagonal components:

$$\text{Tr}(T_{ik}) = \sum_i T_{ii} \quad (\text{X.21})$$

A scalar does not change upon the transformation of coordinates. Consequently, the trace of a tensor is an invariant. For example, the trace of the tensor (X.16) is the scalar product of the vectors \mathbf{a} and \mathbf{b} which, as we know, is invariant relative to the transformation of the coordinates [see the text following formula (VI.28)].

In physical applications, the multiplication of tensors is customarily used in combination with the subsequent contraction of the

¹ For a tensor whose rank r is higher than the second, contraction can be performed in several ways (over different pairs of subscripts). The result will be different tensors of the rank $r - 2$.

expression obtained. The result of these operations is known as the **scalar product of tensors**¹. A typical example is the scalar product of vectors (tensors of the first rank) which we have just mentioned: the tensor $a_i b_k$ is contracted over the pair of subscripts i and k , and the result is the expression $\sum a_k b_k$.

In accordance with the above, by a scalar product of the tensor T_{ik} and the vector a_k is meant the vector b_i with the components

$$b_i = \sum_k T_{ik} a_k \quad (\text{X.22})$$

Let us convince ourselves that the set of quantities b_i determined in this way does form a vector. For this end, let us find the law of transformation of the quantities b_i . It is evident that

$$b'_i = \sum_k T'_{ik} a'_k$$

Let us introduce into the last expression the values of T'_{ik} and a'_k in terms of the unprimed components:

$$b'_i = \sum_k T'_{ik} a'_k = \sum_k \sum_{l,m} \alpha_{il} \alpha_{km} T_{lm} \sum_s \alpha_{ks} a_s = \sum_l \alpha_{il} \sum_{m,s} T_{lm} a_s \sum_k \alpha_{km} \alpha_{ks}$$

According to the property (VI.40), $\sum_k \alpha_{km} \alpha_{ks} = \delta_{ms}$. Consequently,

$$b'_i = \sum_l \alpha_{il} \sum_{m,s} T_{lm} a_s \delta_{ms} = \sum_l \alpha_{il} \sum_m T_{lm} a_m = \sum_l \alpha_{il} b_l$$

The result we have obtained signifies that the quantities b_i are transformed according to the law of transformation of vector components. Hence, the quantities b_i determined by expression (X.22) do indeed form a vector.

In a similar way, we can convince ourselves in the correctness of the following statement: if a set of nine quantities X_{ik} taken with the components of a vector a_k in the combination

$$\sum_k X_{ik} a_k$$

gives the components of another vector b_i , the quantities X_{ik} are the components of a tensor. We encountered such a situation at the beginning of this Appendix. The set of nine quantities χ_{ik} taken with the components of the vector E_k in the combination (X.5) gave the components of the vector P_i . On these premises, we showed that the quantities χ_{ik} are transformed according to the law of transformation of tensor components, i.e. that the dielectric susceptibility is a tensor.

¹ The latter product is sometimes called an inner one, whereas expression (X.20) is called an outer product of tensors.

As a result of the scalar multiplication of the vector \mathbf{a} and the tensor \mathbf{T} , we obtain the new vector \mathbf{b} :

$$\mathbf{b} = \mathbf{T}\mathbf{a} \quad (\text{X.23})$$

The tensor \mathbf{T} can therefore be considered as a linear operator transforming one vector into another one.

Let us find the product of a unit tensor and the vector \mathbf{a}_k . By formula (X.22), we have

$$b_i = \sum_k \delta_{ik} a_k = a_i$$

Hence, a vector does not change when multiplied by a unit tensor.

3. Symmetric and Antisymmetric Tensors. The tensor S_{ik} whose components satisfy the condition

$$S_{ik} = S_{ki} \quad (\text{X.24})$$

is said to be **symmetric**. We must note that the dielectric susceptibility tensor treated above is symmetric.

The tensor A_{ik} whose components satisfy the condition

$$A_{ik} = -A_{ki} \quad (\text{X.25})$$

is said to be **antisymmetric**.

The property of symmetry or antisymmetry belongs to the tensor itself. This follows from the fact that this property is retained in any transformations of the coordinates. We shall give the following calculations to prove this statement:

$$S'_{ik} = \sum_{l,m} \alpha_{il} \alpha_{km} S_{lm} = \sum_{m,l} \alpha_{im} \alpha_{kl} S_{ml} = \sum_{l,m} \alpha_{kl} \alpha_{im} S_{lm} = S'_{ki}$$

(we first interchanged the dummy indices l and m , and then replaced S_{ml} with S_{lm} equal to it). We have proved that a tensor symmetric in the system K will be symmetric in any other system K' .

Similarly

$$A'_{ik} = \sum_{l,m} \alpha_{il} \alpha_{km} A_{lm} = \sum_{m,l} \alpha_{im} \alpha_{kl} A_{ml} = - \sum_{l,m} \alpha_{kl} \alpha_{im} A_{lm} = -A'_{ki}$$

(after interchanging the dummy indices, we replaced A_{ml} with $-A_{lm}$ equal to it). We have proved that a tensor antisymmetric in K will be antisymmetric in K' too.

Any tensor T_{ik} can be written as the sum of a symmetric and an antisymmetric tensors. Indeed, let us write T_{ik} as

$$T_{ik} = \frac{T_{ik} + T_{ki}}{2} + \frac{T_{ik} - T_{ki}}{2}$$

The lawfulness of such an expression is obvious. At the same time, the first term does not change when the subscripts i and k are interchanged, i.e. it has the properties of the components of a symmetric tensor; the second term, on the other hand, changes its sign when

the subscripts are interchanged, i.e. has the properties of the components of an antisymmetric tensor.

We can therefore always consider that

$$T_{ik} = S_{ih} + A_{ih} \quad (\text{X.26})$$

where

$$S_{ih} = \frac{T_{ik} + T_{hi}}{2}, \quad A_{ih} = \frac{T_{ik} - T_{hi}}{2} \quad (\text{X.27})$$

The concepts of symmetry and antisymmetry can also be applied to tensors of a higher rank. For instance, T_{ihl} is said to be symmetric (antisymmetric) relative to the subscripts i and k (or i and l , or k and l) if upon transposition of these subscripts the components of the tensor do not change (reverse their sign). If the components of a tensor do not change (or reverse their sign) upon the transposition of any pair of subscripts, the tensor is said to be **absolutely symmetric** (**absolutely antisymmetric**).

The set of 27 quantities ε_{ihl} introduced in Appendix VI forms an absolutely antisymmetric tensor of the third rank. Recall that the quantities ε_{ihl} (1) are zero if any two subscripts have identical values; (2) equal $+1$ if all the subscripts are different and form a cyclic transposition of the sequence 1, 2, 3; (3) equal -1 if all the subscripts are different and form a cyclic transposition of the sequence 3, 2, 1. Of the 27 components of this tensor, 21 are zero, three are $+1$, and three are -1 . It can be shown by simple but cumbersome calculations that

$$\varepsilon'_{ihl} = \sum_{m, p, s} \alpha_{im} \alpha_{hp} \alpha_{ls} \varepsilon_{mps} = \varepsilon_{ihl} \quad (\text{X.28})$$

i.e. that the tensor ε_{ihl} is invariant.

4. True Tensors and Pseudotensors. Let us see how the components of a tensor transform upon the inversion of the coordinate axes. The transformation coefficients in this case have the values

$$\alpha_{ih} = -\delta_{ih}$$

[see (VI.42)]. Therefore, the formula for the transformation of the components of a tensor of rank r , $T_{ih\dots l}$ (altogether r subscripts), in inversion is

$$\begin{aligned} T'_{ih\dots l} &= \sum_{m, p, \dots s} (-\delta_{im}) (-\delta_{hp}) \dots (-\delta_{ls}) T_{mps\dots} \\ &= (-1)^r \sum_{m, p, \dots s} \delta_{im} \delta_{hp} \dots \delta_{ls} T_{mps\dots} = (-1)^r T_{ih\dots l} \end{aligned}$$

Hence, the components of a tensor of rank r (we have in mind tensors in three-dimensional space) are transformed in inversion by the formula

$$T'_{ih\dots l} = (-1)^r T_{ih\dots l}$$

By (X.28), the quantities ε_{ihl} in any transformation of the coordinates (and, consequently, in inversion) remain invariant, i.e.

do not change their sign. The components of a true tensor of rank three, on the other hand, must reverse their sign in inversion. This is why the set of quantities ε_{ikh} forms a pseudotensor instead of a true one.

A pseudotensor of rank r is defined to be a set of 3^r components $P_{ih\dots l}$ which upon rotations of the coordinate system behave like the components of an ordinary tensor, and upon inversion are transformed by the formula

$$P'_{ih\dots l} = (-1)^{r+1} P_{ih\dots l}$$

Let us consider examples of pseudotensors of different ranks. We acquainted ourselves with pseudoscalars and pseudovectors in Appendix VI. We treated a pseudotensor of the third rank ε_{ikh} above. The set of quantities P_{ih} formed from the products of the components a_i of the true vector and p_h of the pseudovector, i.e.

$$P_{ih} = a_i p_h$$

is a pseudovector of the second rank. Indeed, upon inversion, the quantities a_i change their sign, while the quantities p_h remain unchanged. Therefore, P_{ih} in inversion will change its sign. At the same time, we know that the components of a true tensor of the second rank do not change their sign in inversion. Hence, P_{ih} is a pseudotensor.

It is a simple matter to see that the contraction of a pseudotensor also results in a pseudotensor [the evenness or oddness of a tensor's rank in contraction does not change, while the formulas for the transformation of the components of the initial tensor and its contraction differ by the factor $(-1)^2$]. For instance, in contracting the pseudotensor P_{ih} , we obtain the scalar product of a true vector and a pseudovector, i.e. a pseudoscalar.

The product of a pseudotensor $P_{ih\dots l}$ of any rank and a true tensor $T_{mp\dots s}$ of any rank is a pseudotensor:

$$\Pi_{ih\dots lmp\dots s} = P_{ih\dots l} T_{mp\dots s}$$

To verify this statement, let us compile the following table:

Table X.1

| $P_{ih\dots l}$ | | $T_{mp\dots s}$ | | $\Pi_{ih\dots lmp\dots s}$ | |
|-----------------|---------------------|-----------------|---------------------|----------------------------|---------------------|
| rank | sign upon inversion | rank | sign upon inversion | rank | sign upon inversion |
| even | changes | even | does not change | even | changes |
| even | changes | odd | changes | odd | does not change |
| odd | does not change | even | does not change | odd | does not change |
| odd | does not change | odd | changes | even | changes |

Similarly, we can see that the product of two pseudotensors of any ranks is a true tensor.

5. Properties of a Symmetric Tensor of the Second Rank. Of the nine components of a symmetric tensor of the second rank, only six are independent ($S_{12} = S_{21}$, $S_{13} = S_{31}$, $S_{23} = S_{32}$).

A symmetric tensor of the second rank allows an important geometric interpretation. Before considering it, we shall note that the vector \mathbf{a} can be represented not only by a directed segment of a straight line, but also by a plane whose equation is

$$\mathbf{a}\mathbf{r} = 1 \quad (\text{X.29})$$

where \mathbf{r} is a position vector of a point on a plane (Fig. X.2). Since

$\mathbf{a}\mathbf{r} = ar_a = 1$, we have $r_a = 1/a$. Consequently, Eq. (X.29) defines a plane perpendicular to the vector \mathbf{a} and at a distance $1/a$ from the origin of coordinates. Expression (X.29) can also be written as $a_r r = 1$, whence $r = 1/a_r$. Hence, on the straight line passing through the origin of coordinates and having the direction \mathbf{n} (see Fig. X.2), the plane (X.29) intercepts a segment of the length

$$\rho = \frac{1}{a_n} \quad (\text{X.30})$$

Now let us turn to the symmetric tensor \mathbf{S} . We shall correlate with it a surface determined by the equation

$$\mathbf{r}(\mathbf{S}\mathbf{r}) = 1 \quad (\text{X.31})$$

It is obvious that this surface does not depend on the choice of the coordinate system in which the components of the tensor \mathbf{S} and of the position vector \mathbf{r} are determined. The nature of the surface depends only on the properties of the tensor \mathbf{S} .

Let us choose an arbitrary coordinate system x_1, x_2, x_3 and express the left-hand side of Eq. (X.31) in terms of the components \mathbf{r} and \mathbf{S} in this system. By (X.22), we have

$$(\mathbf{S}\mathbf{r})_i = \sum_h S_{ih} x_h$$

Scalar multiplication of \mathbf{r} and the vector $\mathbf{S}\mathbf{r}$ yields

$$\sum_i x_i (\mathbf{S}\mathbf{r})_i = \sum_i x_i \sum_h S_{ih} x_h = \sum_{i,h} S_{ih} x_i x_h$$

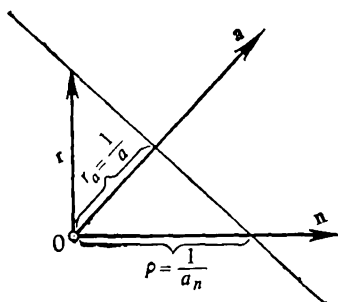


Fig. X.2.

Consequently, Eq. (X.31) has the form

$$\sum_{i,h} S_{ih} x_i x_h = 1 \quad (\text{X.32})$$

In the expanded form, this equation appears as follows (recall that $S_{ih} = S_{hi}$):

$$S_{11}x_1^2 + S_{22}x_2^2 + S_{33}x_3^2 + 2S_{12}x_1x_2 + 2S_{23}x_2x_3 + 2S_{31}x_3x_1 = 1 \quad (\text{X.33})$$

Equation (X.33) determines a second-degree surface whose centre is at the origin of coordinates. In the applications of tensor calculus to physical problems, the diagonal components S_{ii} may be greater than zero (this happens, for example, with the quantities χ_{ii}). In this case, the surface (X.33) is an ellipsoid. It is exactly this ellipsoid that is the geometrical image of a second-rank symmetric tensor (naturally, in three-dimensional space), like a directed line segment or the plane (X.29) is a geometrical image of a first-rank tensor, i.e. a vector.

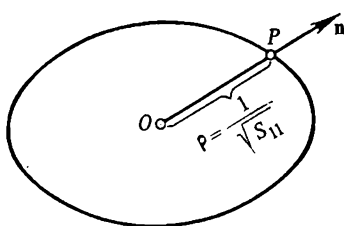


Fig. X.3.

Let us find the distance from the centre of the ellipsoid to points on its surface. To do this, we draw an arbitrary straight line n (Fig. X.3) from the centre of the ellipsoid and assume it to be the x_1 -axis. Now the distance ρ from the centre of the ellipsoid to the point P will equal the value of x_1 when $x_2 = x_3 = 0$. Assuming in Eq. (X.33) that $x_2 = x_3 = 0$, we find that $S_{11}x_1^2 = 1$, whence

$$x_1 = \rho = \frac{1}{\sqrt{S_{11}}}$$

[compare with (X.30)]. Hence, the distance ρ is the reciprocal of the square root of the tensor component S_{11} evaluated for the condition that the direction along which ρ is measured has been taken as the x_1 -axis.

In transformations of the coordinates, the coefficients at S_{ih} in Eq. (X.33) change, but the ellipsoid itself does not depend on the choice of the coordinate system. If we direct the coordinate axes along the semiaxes of the ellipsoid, the equation of the latter, i.e. Eq. (X.33), as is known, becomes simpler and acquires the form

$$S_{11}x_1^2 + S_{22}x_2^2 + S_{33}x_3^2 = 1$$

This signifies that with this choice of the coordinate axes, the non-diagonal tensor components vanish. Consequently, the principal axes of the tensor coincide with the semiaxes of the tensor ellipsoid.

If the coordinate axes are directed along the principal axes of the tensor (i.e. along the semiaxes of the tensor ellipsoid), the tensor acquires a diagonal form:

$$(S_{ik}) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad (\text{X.34})$$

We have introduced the notation $S_{11} = \lambda_1$, $S_{22} = \lambda_2$, $S_{33} = \lambda_3$. The quantities λ_1 , λ_2 , and λ_3 are the principal values of the tensor.

When $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, the tensor ellipsoid transforms into a sphere. Particularly, a sphere of unit radius corresponds to the unit tensor δ_{ik} .

Let us multiply a vector \mathbf{a} by the tensor (X.34). The result is the vector \mathbf{b} whose components are determined by formula (X.22). The components of the tensor (X.34) can be written as $S_{ik} = \lambda_i \delta_{ik}$. Consequently,

$$b_i = \sum_k \lambda_i \delta_{ik} a_k = \lambda_i a_i \quad (\text{X.35})$$

Let the vector \mathbf{a} be directed along the first principal axis of the tensor. Hence, $a_1 = a$, $a_2 = a_3 = 0$ [having taken the tensor in the form of (X.34), we assumed that the coordinate axes are directed along the principal axes of the tensor]. By formula (X.35), the components of the vector \mathbf{b} will be $b_1 = \lambda_1 a_1$, $b_2 = b_3 = 0$. This signifies that the direction of the vector \mathbf{b} coincides with that of the vector \mathbf{a} , while the magnitude of the vector \mathbf{b} is λ_1 times that of the vector \mathbf{a} . The same also holds for the other two principal axes. Therefore, if the vector \mathbf{a} is directed along one of the principal axes of the tensor, the following equation holds:

$$\mathbf{S}\mathbf{a} = \lambda \mathbf{a} \quad (\text{X.36})$$

where λ is the relevant principal value of the tensor.

We have arrived at the following result: when a vector having the direction of one of the principal axes is multiplied by a tensor, the direction of the vector does not change, but its magnitude grows by a number of times equal to the relevant principal value of the tensor.

Equation (X.36) holds for any coordinate system if only the vector \mathbf{a} is directed along one of the principal axes of the tensor. This circumstance can be used to find the principal values and the principal axes of a tensor. For this purpose, we must vary the direction of the vector \mathbf{a} until the vector $\mathbf{S}\mathbf{a}$ coincides in direction with \mathbf{a} . The direction found gives a principal axis of the tensor, and the ratio of $\mathbf{S}\mathbf{a}$ to \mathbf{a} —the corresponding principal value. Analytically, this can be expressed as follows. Let us write Eq. (X.36) in components (we have not yet reduced the tensor to its principal axes so that, generally

speaking, all the S_{ik} 's are non-zero):

$$S_{11}a_1 + S_{12}a_2 + S_{13}a_3 = \lambda a_1$$

$$S_{21}a_1 + S_{22}a_2 + S_{23}a_3 = \lambda a_2$$

$$S_{31}a_1 + S_{32}a_2 + S_{33}a_3 = \lambda a_3$$

Combining like terms, we obtain

$$\left. \begin{aligned} (S_{11} - \lambda)a_1 + S_{12}a_2 + S_{13}a_3 &= 0 \\ S_{21}a_1 + (S_{22} - \lambda)a_2 + S_{23}a_3 &= 0 \\ S_{31}a_1 + S_{32}a_2 + (S_{33} - \lambda)a_3 &= 0 \end{aligned} \right\} \quad (\text{X.37})$$

We have arrived at a homogeneous system of three equations with the unknowns a_1, a_2, a_3 —the components of the required vector \mathbf{a} . For this system to have a non-zero solution, it is necessary that its determinant be zero [see Appendix VIII, the text following formula (VIII.26)]. Hence, the following condition must be satisfied:

$$\begin{vmatrix} S_{11} - \lambda & S_{12} & S_{13} \\ S_{21} & S_{22} - \lambda & S_{23} \\ S_{31} & S_{32} & S_{33} - \lambda \end{vmatrix} = 0 \quad (\text{X.38})$$

The roots of this cubic equation in λ are the principal values of the tensor: $\lambda_1, \lambda_2, \lambda_3$. Introducing one of these roots into the system (X.37), we can find the ratios a_2/a_1 and a_3/a_1 that will determine the direction of the vector \mathbf{a} satisfying the equation (X.36), i.e. the relevant principal axis of the tensor. By performing this operation for all three λ_i 's, we find the directions of all the principal axes.

We have assumed that all the roots λ_1, λ_2 , and λ_3 are different. When Eq. (X.38) has multiple roots, clarification is needed. With two multiple roots ($\lambda_2 = \lambda_3 = \lambda, \lambda_1 \neq \lambda$), the tensor ellipsoid will be an ellipsoid of revolution. Only the principal axis of the tensor coinciding with the axis of symmetry of the ellipsoid will be determined uniquely. Any two mutually perpendicular axes normal to the axis of symmetry of the ellipsoid may be taken as the other two principal axes. If all three roots of Eq. (X.38) are the same ($\lambda_1 = \lambda_2 = \lambda_3 = \lambda$), the tensor ellipsoid degenerates into a sphere, and any three mutually perpendicular axes may be taken as the principal ones.

Let us consider the properties of a symmetric tensor as an operator. By (X.36), the action of \mathbf{S} on the vector \mathbf{a} directed along one of the principal axes of the tensor causes only the length of the vector to change λ times. The action of \mathbf{S} on an arbitrarily oriented vector causes, generally speaking, a change in both the length and the direction of the vector (see, for instance, Fig. X.1).

6. Properties of an Antisymmetric Tensor of the Second Rank.

Let us now turn to an antisymmetric tensor of the second rank. Condition (X.25) for the diagonal components has the form $A_{ii} = -A_{ii}$. This is possible only when $A_{ii} = 0$. Consequently, the diagonal components of an antisymmetric tensor are zero. Of the remaining six components, only three are independent. Hence, an antisymmetric tensor of the second rank, like a vector, is determined by three quantities:

$$\left. \begin{aligned} A_{12} &= -A_{21} = a_3 \\ A_{23} &= -A_{32} = a_2 \\ A_{31} &= -A_{13} = a_1 \end{aligned} \right\} \quad (\text{X.39})$$

Using this notation, we can write an antisymmetric tensor as follows:

$$(A_{ik}) = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \quad (\text{X.40})$$

The components of the tensor (X.40) can be written as

$$A_{ik} = \sum_l \varepsilon_{ikl} a_l = \varepsilon_{ikh1} a_1 + \varepsilon_{ikh2} a_2 + \varepsilon_{ikh3} a_3 \quad (\text{X.41})$$

where ε_{ikl} is an absolutely antisymmetric pseudotensor [see the text preceding formula (X.28)]. Indeed, the components A_{ii} determined by this formula are zero because ε_{iil} at any l equals zero. Further,

$$A_{12} = \varepsilon_{121} a_1 + \varepsilon_{122} a_2 + \varepsilon_{123} a_3 = 0 \cdot a_1 + 0 \cdot a_2 + 1 \cdot a_3 = a_3$$

$$A_{13} = \varepsilon_{131} a_1 + \varepsilon_{132} a_2 + \varepsilon_{133} a_3 = 0 \cdot a_1 + (-1) \cdot a_2 + 0 \cdot a_3 = -a_2$$

etc.

It was shown above that the multiplication of a true tensor by a pseudotensor results in a pseudotensor. The product of two pseudotensors, on the other hand, is a true tensor. The fact that the quantities a_i when multiplied by the pseudotensor ε_{ikl} [see formula (X.41)] yield the components of a true tensor of the second rank indicates that these quantities have the properties of a pseudotensor of the first rank, i.e. a pseudovector.

Consequently, a pseudovector with the components a_i determined by formulas (X.39) can be correlated with any antisymmetric tensor of the second rank. And, conversely, a true antisymmetric tensor of the second rank whose components are expressed in terms of the components of a pseudovector by formula (X.41) can be said to conform to any pseudovector a_i .

Let us apply the operator \mathbf{A} (here \mathbf{A} is an antisymmetric tensor) to a vector \mathbf{b} . By (X.22) and (X.41), we obtain the vector \mathbf{c} having

the components

$$c_i = \sum_k A_{ik} b_k = \sum_{k,l} \varepsilon_{ikl} b_k a_l$$

Comparing the expression obtained with formula (VI.33), we arrive at the conclusion that \mathbf{c} can be written as the vector product¹ of the vectors \mathbf{b} and \mathbf{a} :

$$\mathbf{c} = \mathbf{A}\mathbf{b} = [\mathbf{b}\mathbf{a}]$$

The vector \mathbf{c} is normal both to the initial vector \mathbf{b} and to the pseudovector \mathbf{a} corresponding to the tensor \mathbf{A} . Hence, the action of the tensor \mathbf{A} causes the vector to turn through the angle $\pi/2$. In addition, generally speaking, the length of the vector changes.

XI. Basic Concepts of Vector Analysis

Gradient. Let us consider a scalar field, i.e. a region of space to each of whose points there corresponds a definite value of the scalar φ :

$$\varphi = \varphi(P) = \varphi(\mathbf{r}) = \varphi(x_1, x_2, x_3)$$

where \mathbf{r} is a position vector, and x_1, x_2, x_3 are the Cartesian coordinates of the point P .

An identical value of φ corresponds to all the points of the surface determined by the equation

$$\varphi(x_1, x_2, x_3) = \text{const} \quad (\text{XI.1})$$

A surface having the form of (XI.1) is said to be a constant φ surface. Such a surface can be drawn through any point of a field.

When displaced from the point P over the distance $d\mathbf{r}$, the function φ receives the increment

$$d\varphi = \sum_i \frac{\partial \varphi}{\partial x_i} dx_i$$

The last expression does not depend on the choice of the coordinates x_i , i.e. is an invariant. The set of quantities dx_i forms the vector $d\mathbf{r}$. We can therefore state [see the text following formula (VI.28)] that the quantities $\partial\varphi/\partial x_i$ are the projections of a vector onto the x_i -axis. This vector is called the gradient of the scalar φ and is designated by the symbol $\text{grad } \varphi$. Hence,

$$\text{grad } \varphi = \sum_i \frac{\partial \varphi}{\partial x_i} \mathbf{e}_i \quad (\text{XI.2})$$

or, in conventional notation,

$$\text{grad } \varphi = \frac{\partial \varphi}{\partial x} \mathbf{e}_x + \frac{\partial \varphi}{\partial y} \mathbf{e}_y + \frac{\partial \varphi}{\partial z} \mathbf{e}_z$$

¹ Recall that a vector product is a true vector only if one of the multipliers is a pseudovector. We could also conclude from this circumstance that \mathbf{a} is not a true vector, but a pseudovector.

It is a simple matter to extend the definition (XI.2) to an n -dimensional space. In the latter case, the number of terms in formula (XI.2) will be n instead of three.

We shall prove that the components of a gradient are transformed by formulas (VI.37). Having taken two coordinate systems, K and K' , we can write

$$d\varphi = \sum_i \frac{\partial \varphi}{\partial x_i} dx_i = \sum_i \frac{\partial \varphi}{\partial x'_i} dx'_i \quad (\text{XI.3})$$

Let us express dx_i in terms of dx'_k by formula (VI.38) and introduce these expressions into (XI.3):

$$\sum_i \frac{\partial \varphi}{\partial x_i} \sum_k \alpha_{ki} dx'_k = \sum_i \frac{\partial \varphi}{\partial x'_i} dx'_i$$

We change the sequence of summation over the subscripts i and k on the left-hand side:

$$\sum_k dx'_k \sum_i \alpha_{ki} \frac{\partial \varphi}{\partial x_i} = \sum_i \frac{\partial \varphi}{\partial x'_i} dx'_i$$

The subscripts i and k are dummy ones. We have already noted that a dummy index can be designated by any letter. Therefore, the sum on the left will not change if we interchange the subscripts i and k . The result is

$$\sum_i dx'_i \sum_k \alpha_{ik} \frac{\partial \varphi}{\partial x_k} = \sum_i \frac{\partial \varphi}{\partial x'_i} dx'_i$$

It follows from the relation we have obtained that

$$\frac{\partial \varphi}{\partial x'_i} = \sum_k \alpha_{ik} \frac{\partial \varphi}{\partial x_k}$$

which coincides with (VI.37). We have thus shown that the quantities $\partial \varphi / \partial x_i$ upon transformations of the coordinates behave like the components of a vector.

William Hamilton introduced the vector differential operator ∇ (the del operator or the Hamiltonian operator) that is a vector with the components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$:

$$\nabla = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} = \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \quad (\text{XI.4})$$

The vector ∇ has no meaning by itself. It acquires a meaning when applied to a scalar or vector function. For instance, upon the symbolic multiplication of ∇ by φ , we obtain the gradient of φ :

$$\nabla \varphi = \sum_i \mathbf{e}_i \frac{\partial \varphi}{\partial x_i}$$

Consequently, $\nabla\varphi \equiv \text{grad } \varphi$. By (XI.3), the increment of φ can be written as the scalar product of the vector $\text{grad } \varphi$ and $d\mathbf{r}$:

$$d\varphi = \text{grad } \varphi \cdot d\mathbf{r} = (\nabla\varphi) d\mathbf{r} \quad (\text{XI.5})$$

In movement over a constant φ surface, $d\varphi = 0$. It thus follows in accordance with (XI.5) that the vector $\nabla\varphi$ at each point of the field is directed along a normal to the constant φ surface. Let us find the rate of change in φ along a certain direction l , i.e. $d\varphi/dl$. By (XI.5), the increment of φ over the segment dl is $(\nabla\varphi) dl = (\nabla\varphi)_l dl$, where $(\nabla\varphi)_l$ is the projection of the gradient onto the direction l . Therefore, $d\varphi/dl = [(\nabla\varphi)_l dl]/dl = (\nabla\varphi)_l$. Hence, the projection of the gradient onto a certain direction gives the rate of change in the function in the given direction.

We must note that the vector $\nabla\varphi$ exists at every point of the scalar field φ . Consequently, a gradient forms a vector field, i.e. a region of space to each point of which there corresponds a definite value of the vector $\nabla\varphi$.

Divergence. Assume that we are given the field of the vector \mathbf{a} . The flux of the vector \mathbf{a} through the surface f is defined to be

$$\Phi_a = \int_f a_n df = \int_f \mathbf{a} d\mathbf{f} \quad (\text{XI.6})$$

where a_n is the projection of the vector \mathbf{a} onto a positive normal to the surface element df , and $d\mathbf{f}$ is the vector of the element; its magnitude equals that of the surface element df , while its direction coincides with that of a positive normal to the element. The direction of the positive normal is determined depending on the circumstances. For example, the outward normal is considered to be positive in calculating the flux through a closed surface.

The name "flux" is due to the fact that for the field of a liquid's velocity vector, the integral (XI.6) gives the flux of the liquid through the surface f , i.e. the volume of the liquid flowing through f in unit time.

Let us surround the point P of a field with the closed surface f and evaluate the flux Φ_a through this surface. The ratio of Φ_a to V will characterize the properties of the field in the vicinity of the point P averaged over the volume V confined within f . The smaller the linear dimensions of the volume, the closer will the average characteristic be to the true characteristic of the field at the point P . The scalar quantity

$$\text{div } \mathbf{a} = \lim_{V \rightarrow P} \frac{\Phi_a}{V} = \lim_{V \rightarrow P} \frac{1}{V} \oint a_n df \quad (\text{XI.7})$$

is known as the divergence of the vector field at the point P .

The definition (XI.7) is a most general one not depending on the choice of the coordinate system. Let us find an expression for the divergence in terms of the projections of \mathbf{a} onto the axes of a Cartesian coordinate system. We take a volume in the form of a rectangular parallelepiped with faces perpendicular to the coordinate axes in the vicinity of the point P (Fig. XI.1). Let us find the flux of the vector \mathbf{a} through the faces 1 and 2 perpendicular to the x -axis. An outward normal to the face 1 coincides in direction with the x -axis. Hence, for this face, $a_n = a_{x1}$ (the subscript 1 indicates that the value of a_x is taken at a point on the face 1). An outward normal to the face 2 is opposite in direction to the x -axis. Therefore, we have for it $a_n = -a_{x2}$ (the subscript 2 indicates that the value of a_x is taken at a point on the face 2). The total flux through the faces 1 and 2 is

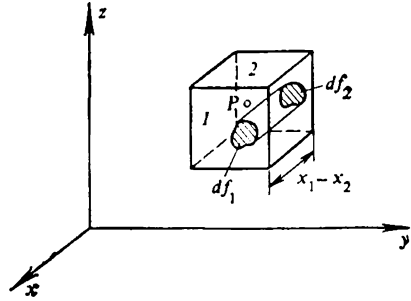


Fig. XI.1.

$$\Phi_x = \int_{(1)} a_{x1} df_1 - \int_{(2)} a_{x2} df_2 = \int_f (a_{x1} - a_{x2}) df \quad (\text{XI.8})$$

where $df = df_1 = df_2$ (see Fig. XI.1), a_{x1} and a_{x2} are taken for points on the faces 1 and 2 with identical y 's and z 's. The integral on the right is taken over the surface f of any of the faces 1 and 2.

Let us expand a_x into a series in the vicinity of the point P :

$$\begin{aligned} a_x = a_{xP} + \left(\frac{\partial a_x}{\partial x} \right)_P (x - x_P) + \left(\frac{\partial a_x}{\partial y} \right)_P (y - y_P) \\ + \left(\frac{\partial a_x}{\partial z} \right)_P (z - z_P) + \varepsilon_x \end{aligned} \quad (\text{XI.9})$$

Here x_P , y_P , and z_P are the coordinates of the point P , a_{xP} is the value of a_x at the point P , $(\partial a_x / \partial x)_P$, etc. are the values of the derivatives at the point P , and ε_x is a quantity of a higher order of smallness than the differences $(x - x_P)$, $(y - y_P)$, and $(z - z_P)$, i.e. a quantity diminishing more rapidly than the linear dimensions of the parallelepiped.

Assuming in expression (XI.9) that $x = x_1$, let us find the values of a_x at points on the face 1, i.e. a_{x1} ; assuming that $x = x_2$, we obtain the values of a_{x2} . Subtracting these values from one another, we obtain the following expression for the opposite areas df_1 and df_2

(the values of y and z are the same for them):

$$a_{x_1} - a_{x_2} = \left(\frac{\partial a_x}{\partial x} \right)_P (x_1 - x_2) + \varepsilon'_x$$

where again ε'_x is a quantity diminishing at a faster rate than the linear dimensions of the volume.

Using the value we have found in formula (XI.8), we obtain

$$\Phi_x = \left(\frac{\partial a_x}{\partial x} \right)_P (x_1 - x_2) \int f df + \int \varepsilon'_x df = \left(\frac{\partial a_x}{\partial x} \right)_P (x_1 - x_2) f + \int \varepsilon'_x df$$

A glance at Fig. XI.1 reveals that the product $(x_1 - x_2) f$ is the volume V of the parallelepiped. Hence,

$$\Phi_x = \left(\frac{\partial a_x}{\partial x} \right)_P V + \varepsilon''_x$$

where ε''_x is a quantity of a higher order of smallness than V . Similar expressions are also obtained for the fluxes through a pair of faces perpendicular to the axes y and z :

$$\Phi_y = \left(\frac{\partial a_y}{\partial y} \right)_P V + \varepsilon''_y, \quad \Phi_z = \left(\frac{\partial a_z}{\partial z} \right)_P V + \varepsilon''_z$$

Summing Φ_x , Φ_y , and Φ_z , we obtain the total flux of the vector \mathbf{a} through the surface of the parallelepiped. By dividing this flux by V in accordance with the definition (XI.7) and performing the limiting process¹ $V \rightarrow 0$, we arrive at the formula

$$\operatorname{div} \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \quad (\text{XI.10})$$

(we have discarded the superfluous subscript P on the derivatives).

Expression (XI.10) we have found for the divergence can be written as

$$\operatorname{div} \mathbf{a} = \sum_i \frac{\partial a_i}{\partial x_i} \quad (\text{XI.11})$$

In this form, the concept of the divergence can be extended to vector fields in an n -dimensional space. The definition (XI.7) can also be extended to an n -dimensional space. In this case, by an element of volume we should understand $dV^* = dx_1 dx_2 dx_3 \dots dx_n$. The integral must be evaluated over an $(n - 1)$ -dimensional hypersurface. An element of the hypersurface perpendicular to the x_k -axis will equal $df^* = dx_1 dx_2 \dots dx_{k-1} dx_{k+1} \dots dx_n$. In a four-dimensional space, a conventional three-dimensional volume will be the hypersurface.

¹ In the limiting process, the quantities ε''/V shrink to zero.

Expression (XI.11) can be considered as the sum of products of the quantities $\nabla_i = \partial/\partial x_i$ and a_i , i.e. as the scalar product of the vectors ∇ and \mathbf{a} . The divergence can therefore be written as

$$\operatorname{div} \mathbf{a} \equiv \nabla \mathbf{a} \quad (\text{XI.12})$$

The quantity $\nabla \mathbf{a}$ exists at every point of the vector field of \mathbf{a} . Consequently, the divergence forms a scalar field determined in the same part of space as the field of the vector \mathbf{a} .

Let us take the finite volume V confined within the surface f (Fig. XI.2) in the field of the vector \mathbf{a} . We divide this volume into elementary volumes ΔV . By (XI.7) for the flux $\Delta \Phi_a$ through the surface of such a volume, we can write $\Delta \Phi_a \approx \operatorname{div} \mathbf{a} \cdot \Delta V = \nabla \mathbf{a} \cdot \Delta V$. Let us summate these expressions for all the elementary volumes. In summing $\Delta \Phi_a$, the fluxes through the faces separating two adjacent volumes mutually nullify each other (for adjacent volumes, the fluxes differ in their signs because the outward normals \mathbf{n} and \mathbf{n}' have opposite directions). Only the fluxes through the portions of the outer surface f remain uncompensated, so that the sum gives us the flux of the vector \mathbf{a} through this surface. The sum on the right in the limit (when $\Delta V \rightarrow 0$) transforms into an integral over the entire volume. The approximate equality in the limit will transform into a strict equality. As a result, we obtain

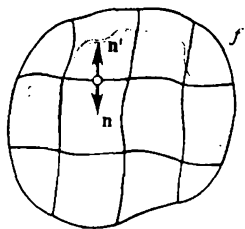


Fig. XI.2.

$$\oint_{f'} \mathbf{a} \, df = \int_V \nabla \mathbf{a} \, dV \quad (\text{XI.13})$$

The relation we have obtained is called the *Ostrogradsky-Gauss theorem*. It states that *the flux of a vector through a closed surface equals the integral of the divergence over the volume confined by this surface*.

Curl. The circulation of the vector \mathbf{a} over the contour Γ is defined to be the expression

$$C_a = \oint_{\Gamma} \mathbf{a} \, d\mathbf{l} = \oint_{\Gamma} a_i \, dl \quad (\text{XI.14})$$

For example, in a potential field of forces, the circulation of the vector \mathbf{F} equals the work of the forces on the closed path Γ .

Let us take in the vicinity of the point P the contour Γ in a plane passing through P . We shall find the circulation C_a around this contour. The ratio of C_a to the surface f enclosed by the contour will characterize the properties of the field in the vicinity of the point P , averaged over the surface f . The smaller the linear dimensions of the surface, the closer will the average characteristic be to the true

characteristic of the field at the point P . At the limit, when the contour shrinks to the point P , the average characteristic will transform into the true one. Therefore, the properties of the vector field at the point P can be characterized by the quantity

$$\lim_{f \rightarrow 0} \frac{C_a}{f} = \lim_{f \rightarrow 0} \frac{1}{f} \oint_{\Gamma} \mathbf{a} \, dl \quad (\text{XI.15})$$

The quantity (XI.15) depends not only on the properties of the field at the point P , but also on the orientation of the plane in which the contour is. The orientation of this plane in space can be set by a normal to the plane associated with the direction of circumvention of the contour Γ in integration by the right-hand screw rule (Fig. XI.3). For different directions of \mathbf{n} , the quantity (XI.15) will have different values at the same point P , and it is not difficult to see that values of the quantity (XI.15) differing only in their sign will correspond to opposite directions of \mathbf{n} . Consequently, the quantity defined by formula (XI.15) behaves like the projection of a vector onto the direction

of a normal to the contour Γ . This vector is known as the **curl** of the vector field at the point P and is designated by the symbol $\text{curl } \mathbf{a}$. Hence,

$$(\text{curl } \mathbf{a})_n = \lim_{f \rightarrow 0} \frac{C_a}{f} = \lim_{f \rightarrow 0} \frac{1}{f} \oint_{\Gamma} \mathbf{a} \, dl \quad (\text{XI.16})$$

Formula (XI.16) gives a most general definition of the curl that does not depend on the choice of the coordinate system. Let us find an expression for the curl in terms of the projections of the vector \mathbf{a} onto the axes of a Cartesian coordinate system. We shall begin with determination of the projection of the vector $\text{curl } \mathbf{a}$ onto the x -axis. For this purpose, we take in the vicinity of the point P the contour Γ in a plane perpendicular to the x -axis (Fig. XI.4). We choose the direction of circumvention of the contour so that it forms a right-handed system with the direction of the x -axis. Hence, the directions of \mathbf{n} and the x -axis will coincide, and expression (XI.16) will give $(\text{curl } \mathbf{a})_x$. For the contour we have chosen, we have

$$\oint_{\Gamma} \mathbf{a}_l \, dl = \oint_{\Gamma} \mathbf{a} \, dl = \oint_{\Gamma} (a_y \, dy + a_z \, dz)$$

(for all the elements of the contour $dx = 0$).

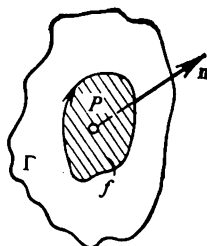


Fig. XI.3.

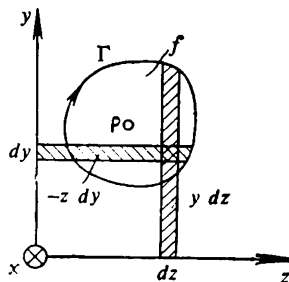


Fig. XI.4.

The values of a_y for points of the contour can be written as¹

$$\begin{aligned} a_y &= a_{yP} + \left(\frac{\partial a_y}{\partial y} \right)_P (y - y_P) + \left(\frac{\partial a_y}{\partial z} \right)_P (z - z_P) + \varepsilon_y \\ &= \text{const} + \left(\frac{\partial a_y}{\partial y} \right)_P y + \left(\frac{\partial a_y}{\partial z} \right)_P z + \varepsilon_y \end{aligned}$$

where the constant term includes three addends not depending on y and z , and ε_y is a quantity of a higher order of smallness than the linear dimensions of the contour. Consequently,

$$\oint a_y dy = \text{const} \oint dy + \left(\frac{\partial a_y}{\partial y} \right)_P \oint y dy + \left(\frac{\partial a_y}{\partial z} \right)_P \oint z dy + \oint \varepsilon_y dy$$

It is not difficult to see that $\oint dy = 0$. In exactly the same way, the integral $\oint y dy = \frac{1}{2} \oint d(y^2)$ is zero. It is simple to see from Fig. XI.4 that $\oint z dy = -f$, where f is the area of the contour. Hence,

$$\oint a_y dy = - \left(\frac{\partial a_y}{\partial z} \right)_P f + \varepsilon'_y \quad (\text{XI.17})$$

where ε'_y is a quantity of a higher order of smallness than the area f of the contour.

Similar calculations for a_z yield the expression

$$\oint a_z dz = \text{const} \oint dz + \left(\frac{\partial a_z}{\partial y} \right)_P \oint y dz + \left(\frac{\partial a_z}{\partial z} \right)_P \oint z dz + \oint \varepsilon_z dz$$

The integrals $\oint dz$ and $\oint z dz = \frac{1}{2} \oint d(z^2)$ are zero, and $\oint y dz = f$. Therefore,

$$\oint a_z dz = \left(\frac{\partial a_z}{\partial y} \right)_P f + \varepsilon'_z \quad (\text{XI.18})$$

The sum of expressions (XI.17) and (XI.18) yields $\oint \mathbf{a} d\mathbf{l}$. Dividing this sum by f in accordance with the definition (XI.16) and performing a limiting process², we obtain

$$(\text{curl } \mathbf{a})_x = \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}$$

Having considered the circulation for contours oriented with the normal \mathbf{n} along the y - and z -axes, we can obtain expressions for the

¹ The meaning of the quantities in this expression is similar to the meaning of the ones in formula (XI.9); the term containing $(x - x_P)$ is absent because $x = x_P$ for all the points of the contour.

² The quantities ε'/f vanish in a limiting process.

projections of the curl onto these axes:

$$(\text{curl } \mathbf{a})_y = \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}, \quad (\text{curl } \mathbf{a})_z = \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}.$$

It is easy to remember the formulas for the projections of a curl onto the coordinate axes by giving attention to the fact that in each of them the subscript on curl \mathbf{a} and the letters on the right in the denominators form a cyclic transposition following the scheme



Knowing the projections, it is simple to find the vector itself:

$$\text{curl } \mathbf{a} = \mathbf{e}_x \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) + \mathbf{e}_y \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) + \mathbf{e}_z \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \quad (\text{XI.19})$$

With a view to the fact that, for example, $\partial a_x / \partial y$ can be written as $\nabla_y a_x$, etc. ($\nabla_y = \partial / \partial y$ is the projection of the vector ∇ onto the y -axis), formula (XI.19) becomes

$$\text{curl } \mathbf{a} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \quad (\text{XI.20})$$

Finally, a comparison with formula (VI.31) allows us to write that

$$\text{curl } \mathbf{a} \equiv [\nabla \mathbf{a}] \quad (\text{XI.21})$$

Taking advantage of formulas (VI.29) and (VI.33) for a vector product, we can write the following expressions for a curl and its k -th component:

$$[\nabla \mathbf{a}] = \sum_{i, k, l} \varepsilon_{ikl} \frac{\partial a_k}{\partial x_l} \mathbf{e}_i, \quad [\nabla \mathbf{a}]_k = \sum_{m, n} \varepsilon_{kmn} \frac{\partial a_n}{\partial x_m} \quad (\text{XI.22})$$

The quantity $[\nabla \mathbf{a}]$ exists at every point of the vector field of \mathbf{a} . Consequently, the curl forms a vector field determined in the same part of space in which the field of the vector \mathbf{a} is set.

Let us take an arbitrary surface f enclosed by the contour Γ (the latter does not necessarily have to be plane and can have any shape) in the field of the vector \mathbf{a} . We divide this surface into small elements Δf (Fig. XI.5). By (XI.16) for the circulation ΔC_a around the boundary of an element Δf , we can write the expression $\Delta C_a \approx [\nabla \mathbf{a}]_n \Delta f$, where $[\nabla \mathbf{a}]_n$ is the projection of $[\nabla \mathbf{a}]$ onto a normal to the given element Δf related to the direction of circumvention by the right-hand screw rule.

Let us summate these expressions for all the Δf 's. In the summation of ΔC_a , the integrals $\int \mathbf{a} \, dl$ taken along the boundaries between adjacent areas cancel each other (for adjacent areas these integrals differ in their sign because they are evaluated in opposite directions). Only the integrals $\int \mathbf{a} \, dl$ for the sections coinciding with the contour Γ enclosing f remain uncompensated. The sum of these integrals gives the circulation of \mathbf{a} around the contour Γ . The sum on the right in the limit (when $\Delta f \rightarrow 0$) transforms into an integral over the surface. The approximate equality in the limit transforms into a strict equality. The result is

$$\oint_{\Gamma} \mathbf{a} \, dl = \int_f [\nabla \mathbf{a}] \, df \quad (\text{XI.23})$$

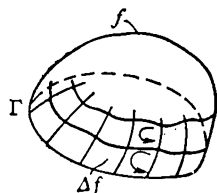


Fig. XI. 5.

The relation we have found is called Stokes's theorem. It states that *the circulation of the vector \mathbf{a} around the closed contour Γ equals the flux of the vector $[\nabla \mathbf{a}]$ through the surface enclosed by the contour Γ .*

The surface over which the integral on the right-hand side of formula (XI.23) is evaluated may be any one, it is only important that it bound on the contour Γ . The direction of the normal \mathbf{n} must be made to agree with the direction of circumventing the contour Γ in integration.

Application of the Operator ∇ to the Product of Functions. When formulas including ∇ are being compiled, it is necessary to adhere to the rules of both vector algebra and differential calculus. Assume, for instance, that φ and ψ are scalar position functions. Therefore,

$$\nabla (\varphi\psi) = \nabla_{\psi} (\varphi\psi) + \nabla_{\varphi} (\varphi\psi) \quad (\text{XI.24})$$

(the subscript on ∇ indicates which of the functions it is acting on). The factor on which ∇ does not act in the given term can be put outside the symbol ∇ (the operator ∇ acts only on quantities following it). Formula (XI.24) now becomes $\nabla (\varphi\psi) = \varphi \nabla_{\psi} \psi + \psi \nabla_{\varphi} \varphi$. In this expression, the subscripts on ∇ are superfluous, so that we finally have

$$\nabla (\varphi\psi) = \varphi \nabla \psi + \psi \nabla \varphi \quad (\text{XI.25})$$

(the right-hand side is read "phi gradient of psi plus psi gradient of phi").

Let us apply ∇ to the product $\varphi \mathbf{a}$. Here there are two possibilities—a scalar or a vector product of the vectors ∇ and $\varphi \mathbf{a}$. The corresponding results are:

$$\nabla (\varphi \mathbf{a}) = \nabla_{\varphi} (\varphi \mathbf{a}) + \nabla_{\mathbf{a}} (\varphi \mathbf{a}) = \mathbf{a} \nabla \varphi + \varphi \nabla \mathbf{a} \quad (\text{XI.26})$$

("a gradient of phi plus phi divergence of a") and

$$[\nabla, (\varphi \mathbf{a})] = [\nabla_\varphi, (\varphi \mathbf{a})] + [\nabla_a, (\varphi \mathbf{a})] = [(\nabla \varphi), \mathbf{a}] + \varphi [\nabla \mathbf{a}] \quad (\text{XI.27})$$

Now let us apply ∇ to the product $[\mathbf{a}\mathbf{b}]$, first obtaining a scalar product of the vectors: $\nabla [\mathbf{a}\mathbf{b}] = \nabla_a [\mathbf{a}\mathbf{b}] + \nabla_b [\mathbf{a}\mathbf{b}]$. We perform the cyclic transposition (VI.3) in each of the terms:

$$\nabla [\mathbf{a}\mathbf{b}] = \mathbf{b} [\nabla_a \mathbf{a}] + \mathbf{a} [\mathbf{b} \nabla_b] = \mathbf{b} [\nabla_a \mathbf{a}] - \mathbf{a} [\nabla_b \mathbf{b}]$$

(we have exchanged the places of \mathbf{b} and ∇_b in the second term to have the vector \mathbf{b} following the operator ∇_b that acts on it; the sign in the vector product has changed accordingly). Omitting the subscripts needed no longer, we arrive at the formula

$$\nabla [\mathbf{a}\mathbf{b}] = \mathbf{b} [\nabla \mathbf{a}] - \mathbf{a} [\nabla \mathbf{b}] \quad (\text{XI.28})$$

("b curl of a minus a curl of b").

The vector product of $[\mathbf{a}\mathbf{b}]$ by ∇ yields $[\nabla, [\mathbf{a}\mathbf{b}]] = [\nabla_a, [\mathbf{a}\mathbf{b}]] + [\nabla_b, [\mathbf{a}\mathbf{b}]]$. Let us expand each of the terms using formula (VI.5): $[\nabla, [\mathbf{a}\mathbf{b}]] = \mathbf{a} (\nabla_a \mathbf{b}) - \mathbf{b} (\nabla_a \mathbf{a}) + \mathbf{a} (\nabla_b \mathbf{b}) - \mathbf{b} (\nabla_b \mathbf{a})$. Arranging the factors so that we may suppress the subscripts on ∇ , we obtain

$$[\nabla, [\mathbf{a}\mathbf{b}]] = (\mathbf{b} \nabla) \mathbf{a} - (\mathbf{a} \nabla) \mathbf{b} + \mathbf{a} (\nabla \mathbf{b}) - \mathbf{b} (\nabla \mathbf{a}) \quad (\text{XI.29})$$

The expressions $(\mathbf{a} \nabla)$ and $(\mathbf{b} \nabla)$ are scalar differential operators. For example,

$$(\mathbf{a} \nabla) = a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} = \sum_i a_i \frac{\partial}{\partial x_i} \quad (\text{XI.30})$$

These operators can be applied to both scalar and vector functions. When applied to the scalar φ , the operator (XI.30) gives

$$(\mathbf{a} \nabla) \varphi = \sum_i a_i \frac{\partial \varphi}{\partial x_i} = \mathbf{a} (\nabla \varphi) \quad (\text{XI.31})$$

The action of the operator $(\mathbf{a} \nabla)$ on the vector \mathbf{b} yields the expression

$$(\mathbf{a} \nabla) \cdot \mathbf{b} = \sum_i a_i \frac{\partial \mathbf{b}}{\partial x_i} = \sum_i a_i \frac{\partial}{\partial x_i} \left(\sum_k \mathbf{e}_k b_k \right) = \sum_k \mathbf{e}_k \sum_i a_i \frac{\partial b_k}{\partial x_i} \quad (\text{XI.32})$$

Let us apply the operator (XI.30) to the product of the scalar function φ and the vector function \mathbf{b} :

$$\begin{aligned} (\mathbf{a} \nabla) (\varphi \mathbf{b}) &= (\mathbf{a} \nabla_\varphi) (\varphi \mathbf{b}) + (\mathbf{a} \nabla_b) (\varphi \mathbf{b}) = \mathbf{b} (\mathbf{a} \nabla) \varphi + \varphi (\mathbf{a} \nabla) \mathbf{b} \\ &= \mathbf{b} (\mathbf{a} \cdot \nabla \varphi) + \varphi (\mathbf{a} \nabla) \mathbf{b} \end{aligned} \quad (\text{XI.33})$$

It is useful to know the value of the expression $(\mathbf{a} \nabla) \mathbf{r}$, where \mathbf{r} is a position vector, and \mathbf{a} is an arbitrary vector. Substituting \mathbf{r} for \mathbf{b} in (XI.32) and taking into account that $\partial x_k / \partial x_i = \delta_{ik}$, we

obtain

$$(\mathbf{a}\nabla)\mathbf{r} = \sum_k \mathbf{e}_k \sum_i a_i \delta_{ik} = \sum_k \mathbf{e}_k a_k = \mathbf{a} \quad (\text{XI.34})$$

We have obtained formulas (XI.25)-(XI.29) quite easily. The finding of the gradient of the scalar product of the two vectors $\nabla(\mathbf{a}\mathbf{b})$ is more involved because it is not clear what should be understood, for example, by the expression $\nabla_a(\mathbf{a}\mathbf{b})$. It cannot be interpreted as $(\nabla_a \mathbf{a})\mathbf{b}$ because the operations of multiplying \mathbf{a} and \mathbf{b} and of applying ∇_a cannot be interchanged. This difficulty can be surmounted by employing auxiliary relations following from formula (VI.5): $[\mathbf{a}, [\nabla\mathbf{b}]] = \nabla_b(\mathbf{a}\mathbf{b}) - \mathbf{b}(\nabla_b \mathbf{a}) = \nabla_b(\mathbf{a}\mathbf{b}) - (\mathbf{a}\nabla)\mathbf{b}$, whence

$$\nabla_b(\mathbf{a}\mathbf{b}) = [\mathbf{a}, [\nabla\mathbf{b}]] + (\mathbf{a}\nabla)\mathbf{b} \quad (\text{XI.35})$$

Having written $[\mathbf{b}, [\nabla\mathbf{a}]]$ in the same way, we arrive at the relation

$$\nabla_a(\mathbf{a}\mathbf{b}) = [\mathbf{b}, [\nabla\mathbf{a}]] + (\mathbf{b}\nabla)\mathbf{a} \quad (\text{XI.36})$$

Substitution of relations (XI.35) and (XI.36) into the formula

$$\nabla(\mathbf{a}\mathbf{b}) = \nabla_a(\mathbf{a}\mathbf{b}) + \nabla_b(\mathbf{a}\mathbf{b})$$

yields the following expression for the gradient of the scalar product of the vectors \mathbf{a} and \mathbf{b} :

$$\nabla(\mathbf{a}\mathbf{b}) = [\mathbf{a}, [\nabla\mathbf{b}]] + [\mathbf{b}, [\nabla\mathbf{a}]] + (\mathbf{a}\nabla)\mathbf{b} + (\mathbf{b}\nabla)\mathbf{a} \quad (\text{XI.37})$$

Repeated Application of the Operator ∇ . The action of the del operator on scalar or vector functions results in new vector or scalar functions which the operator ∇ can be applied to, in turn.

The gradient of the function φ is a vector. Consequently, both the divergence and the curl operations can be applied to it. Let us calculate the divergence of a gradient. By formulas (XI.2) and (XI.11), we have

$$\nabla(\nabla\varphi) = \nabla^2\varphi = \sum_i \frac{\partial}{\partial x_i} (\nabla\varphi)_i = \sum_i \frac{\partial}{\partial x_i} \frac{\partial\varphi}{\partial x_i} = \sum_i \frac{\partial^2\varphi}{\partial x_i^2} = \Delta\varphi$$

Hence

$$\nabla(\nabla\varphi) = \Delta\varphi \quad (\text{XI.38})$$

where Δ is the Laplacian operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (\text{XI.39})$$

Our calculations show that

$$\nabla^2 = \Delta \quad (\text{XI.40})$$

It must be borne in mind, however, that such a relation between the operators ∇ and Δ occurs only in Cartesian coordinates. In other coordinate systems, for instance in a cylindrical or a spherical one.

relation (XI.40) is not observed. The most general definition of the operator Δ that holds in any coordinate system is that following from relation (XI.38), which can be written as

$$\Delta\varphi = \operatorname{div} \operatorname{grad} \varphi \quad (\text{XI.41})$$

Let us find $\operatorname{curl} \operatorname{grad} \varphi$. By (XI.22), we have

$$[\nabla, \nabla\varphi] = \sum_{i, k, l} \varepsilon_{ikl} \frac{\partial}{\partial x_i} \left(\frac{\partial\varphi}{\partial x_k} \right) \mathbf{e}_l \quad (\text{XI.42})$$

Since $\partial^2\varphi/\partial x_i \partial x_k = \partial^2\varphi/\partial x_k \partial x_i$, expression (XI.42) is zero, so that

$$\operatorname{curl} \operatorname{grad} \varphi = 0 \quad (\text{XI.43})$$

This is what should be expected because $[\nabla, \nabla\varphi] = [\nabla\nabla] \varphi$, and the vector product of a vector by itself is zero.

Let us calculate the divergence of a curl. By formulas (XI.11) and (XI.22), we obtain

$$\nabla[\nabla\mathbf{a}] = \sum_k \frac{\partial}{\partial x_k} \left(\sum_{m, n} \varepsilon_{kmn} \frac{\partial a_n}{\partial x_m} \right) = \sum_{k, m, n} \varepsilon_{kmn} \frac{\partial^2 a_n}{\partial x_k \partial x_m}$$

Seeing that $\partial^2 a_n / \partial x_k \partial x_m = \partial^2 a_n / \partial x_m \partial x_k$, the last expression equals zero. Consequently,

$$\operatorname{div} \operatorname{curl} \mathbf{a} = 0 \quad (\text{XI.44})$$

We could have arrived at this result directly by taking into account that the scalar triple product of vectors (which is what $\nabla[\nabla\mathbf{a}]$ is) equals the volume of a parallelepiped constructed on the vectors being multiplied. Therefore, when two of the three factors coincide, such a product is zero.

To calculate $\operatorname{curl} \operatorname{curl} \mathbf{a}$, we shall proceed from formulas (XI.22):

$$[\nabla, [\nabla\mathbf{a}]] = \sum_{i, k, l} \varepsilon_{ikl} \frac{\partial}{\partial x_i} \left(\sum_{m, n} \varepsilon_{kmn} \frac{\partial a_n}{\partial x_m} \right) \mathbf{e}_l$$

Let us perform a cyclic transposition of the subscripts on ε so that the subscript k is the last on both ε 's, and let us use relation (VI.16). The result is

$$\begin{aligned} [\nabla, [\nabla\mathbf{a}]] &= \sum_{i, l, m, n} \mathbf{e}_l \frac{\partial^2 a_n}{\partial x_i \partial x_m} \sum_k \varepsilon_{litk} \varepsilon_{mnk} \\ &= \sum_{i, l, m, n} \mathbf{e}_l \frac{\partial^2 a_n}{\partial x_i \partial x_m} (\delta_{lm} \delta_{in} - \delta_{ln} \delta_{im}) \\ &= \sum_{i, l, m, n} \mathbf{e}_l \frac{\partial^2 a_n}{\partial x_i \partial x_m} \delta_{lm} \delta_{in} - \sum_{i, l, m, n} \mathbf{e}_l \frac{\partial^2 a_n}{\partial x_i \partial x_m} \delta_{ln} \delta_{im} \end{aligned}$$

We summate over the subscripts m and n . This causes the expression we have obtained to become

$$\sum_{i,l} e_l \frac{\partial^2 a_l}{\partial x_l \partial x_l} - \sum_{i,l} e_l \frac{\partial^2 a_l}{\partial x_l^2}$$

which can be written as follows:

$$\sum_l e_l \frac{\partial}{\partial x_l} \left(\sum_i \frac{\partial a_l}{\partial x_l} \right) - \sum_l \frac{\partial^2}{\partial x_l^2} \left(\sum_{i,l} a_l e_l \right) = \nabla (\nabla a) - \Delta a$$

We have thus arrived at the formula

$$[\nabla, [\nabla a]] = \nabla (\nabla a) - \Delta a \quad (\text{XI.45})$$

or

$$\text{curl curl } a = \text{grad div } a - \Delta a \quad (\text{XI.46})$$

It is a simple matter to see that relation (XI.45) can be obtained if we expand $[\nabla, [\nabla a]]$ by formula (VI.5), treating ∇ as a conventional vector.

Examination of (XI.46) shows that

$$\text{grad div } a = \text{curl curl } a + \Delta a \quad (\text{XI.47})$$

The divergence is a scalar. Therefore, no operation except that of finding the gradient can be applied to it.

Some Formulas of Vector Analysis. Let us find the divergence and curl of the position vector \mathbf{r} , and also the gradient of the magnitude of \mathbf{r} . Taking into account that $r = (\sum x_i^2)^{1/2}$, we obtain by (XI.2)

$$\nabla r = \sum_i e_i \frac{\partial r}{\partial x_i} = \sum_i e_i \frac{x_i}{r} = \frac{\mathbf{r}}{r} = \mathbf{e}_r \quad (\text{XI.48})$$

where \mathbf{e}_r is the unit vector of the position vector \mathbf{r} .

By (XI.11), we have

$$\nabla r = \sum_i \frac{\partial r_l}{\partial x_l} = \sum_l \frac{\partial x_l}{\partial x_l} = 3 \quad (\text{XI.49})$$

(we have taken advantage of the fact that $r_l = x_l$). In an n -dimensional space, $\nabla r = n$.

It is not difficult to see that the curl of the position vector is zero:

$$[\nabla \mathbf{r}] = 0 \quad (\text{XI.50})$$

For this end, we can use formula (XI.22), substituting x_k for a_k in it. We obtain

$$[\nabla \mathbf{r}] = \sum_{i,k,l} \varepsilon_{ikh} \frac{\partial x_k}{\partial x_l} \mathbf{e}_l = \sum_{i,k,l} \varepsilon_{ikh} \delta_{lk} \mathbf{e}_l$$

This expression is zero because at identical values of the subscripts

i and k , the factor ε_{ikh} vanishes, and at different values of them, the factor δ_{ik} vanishes.

Relation (XI.50) also follows from the fact that the circulation of the vector \mathbf{r} around any contour is zero. Indeed, according to (XI.14)

$$C_r = \oint \mathbf{r} d\mathbf{l} = \oint (x dx + y dy + z dz) = \frac{1}{2} \oint d(x^2 + y^2 + z^2)$$

There is a total differential in the integrand, therefore the latter is zero.

Now let us find the gradient of a function of r , i.e. $\nabla\varphi(r)$. The partial derivative of $\varphi(r)$ with respect to x_k has the form¹ $\partial\varphi/\partial x_k = \partial\varphi/\partial r \cdot \partial r/\partial x_k = \partial\varphi/\partial r \cdot x_k/r$. Consequently,

$$\nabla\varphi(r) = \sum_k \mathbf{e}_k \frac{\partial\varphi}{\partial r} \frac{x_k}{r} = \frac{\partial\varphi}{\partial r} \frac{1}{r} \sum_k \mathbf{e}_k x_k = \frac{\partial\varphi}{\partial r} \frac{\mathbf{r}}{r}$$

Taking formula (XI.48) into consideration, we can write

$$\nabla\varphi(r) = \frac{\partial\varphi}{\partial r} \nabla r = \frac{\partial\varphi}{\partial r} \frac{\mathbf{r}}{r} = \frac{\partial\varphi}{\partial r} \mathbf{e}_r \quad (\text{XI.51})$$

Assume that we have a function of the distance between two points:

$$\varphi(|\mathbf{r} - \mathbf{r}'|) = \varphi\left(\sqrt{\sum_i (x_i - x'_i)^2}\right) = \varphi(R), \text{ where } R = \sqrt{\sum_i (x_i - x'_i)^2}$$

The operation of finding the gradient can be applied to this function in two ways—we can perform differentiation either with respect to the coordinates x_i or to the coordinates x'_i . To distinguish between these two gradients, in the first case we shall use the symbol ∇ , and in the second, ∇' . The components of the gradient in both cases have the form

$$(\nabla\varphi)_i = \frac{\partial\varphi}{\partial R} \frac{\partial R}{\partial x_i}, \quad (\nabla'\varphi)_i = \frac{\partial\varphi}{\partial R} \frac{\partial R}{\partial x'_i}$$

It is evident that the derivatives $\partial R/\partial x_i$ and $\partial R/\partial x'_i$ differ only in their sign. We thus conclude that

$$\nabla\varphi = -\nabla'\varphi \quad (\text{XI.52})$$

Let us calculate the curl of the unit vector $\mathbf{e}_r = \mathbf{r}/r$. Representing \mathbf{r}/r as the product of $\varphi = 1/r$ and $\mathbf{a} = \mathbf{r}$, we shall apply formula (XI.27). As a result, we find that $[\nabla, \mathbf{r}/r] = [\nabla(1/r), \mathbf{r}] + (1/r)[\nabla, \mathbf{r}]$. By (XI.51), we have $\nabla(1/r) = -(1/r^2)\mathbf{r}/r$. Therefore, the first term is zero. The second term is zero in accordance with (XI.50). Hence,

$$[\nabla, \mathbf{r}/r] = [\nabla, \mathbf{e}_r] = 0 \quad (\text{XI.53})$$

¹ We have written $\partial\varphi/\partial r$ instead of $d\varphi/dr$ because it may happen that φ , in addition to r , also depends, say, on t .

Now let us find the curl of the function $\varphi(r) \mathbf{e}_r$, where $\varphi(r)$ is an arbitrary function of r . We again apply formulas (XI.27) and (XI.51):

$$\begin{aligned} [\nabla, \varphi(r) \mathbf{e}_r] &= [\nabla \varphi(r), \mathbf{e}_r] + \varphi(r) [\nabla, \mathbf{e}_r] \\ &= [(\partial \varphi / \partial r) \mathbf{e}_r, \mathbf{e}_r] + \varphi(r) [\nabla, \mathbf{e}_r] = 0 \end{aligned}$$

[see (XI.53)]. Hence

$$[\nabla, \varphi(r) \mathbf{e}_r] = 0 \quad (\text{XI.54})$$

Assume that we are given the vector function $\mathbf{a}(\xi) = \sum_k a_k(\xi) \mathbf{e}_k$ of the scalar quantity ξ that, in turn, is a function of the coordinates x_i , that is, $\xi = \xi(x_i)$. Let us find the curl and divergence of this function. According to the rules for the differentiation of a composite function,

$$\frac{\partial a_k}{\partial x_i} = \frac{\partial a_k}{\partial \xi} \frac{\partial \xi}{\partial x_i} \quad (\text{XI.55})$$

By (XI.22)

$$[\nabla \mathbf{a}] = \sum_{i, k, l} \varepsilon_{ikl} \frac{\partial a_k}{\partial x_i} \mathbf{e}_l = \sum_{i, k, l} \varepsilon_{ikl} \frac{\partial a_k}{\partial \xi} \frac{\partial \xi}{\partial x_i} \mathbf{e}_l$$

The quantity $\partial a_k / \partial \xi$ is the k -th component of the vector $\partial \mathbf{a} / \partial \xi$, and $\partial \xi / \partial x_i$ is the i -th component of the gradient of the function ξ . Therefore,

$$[\nabla \mathbf{a}] = \sum_{i, k, l} \varepsilon_{ikl} (\nabla \xi)_i \left(\frac{\partial \mathbf{a}}{\partial \xi} \right)_k \mathbf{e}_l = \left[\nabla \xi, \frac{\partial \mathbf{a}}{\partial \xi} \right]$$

[we have used formula (VI.29)]. Hence, if $\mathbf{a} = \mathbf{a}(\xi)$, where $\xi = \xi(x_1, x_2, x_3)$, we have

$$[\nabla \mathbf{a}] = \left[\nabla \xi, \frac{\partial \mathbf{a}}{\partial \xi} \right] \quad (\text{XI.56})$$

To find the divergence of the function $\mathbf{a}(\xi)$, we proceed from formula (XI.11). With a view to (XI.55), we obtain

$$\nabla \mathbf{a} = \sum_k \frac{\partial a_k}{\partial x_k} \mathbf{e}_k = \sum_k \frac{\partial a_k}{\partial \xi} \frac{\partial \xi}{\partial x_k} \mathbf{e}_k = \sum_k \left(\frac{\partial \mathbf{a}}{\partial \xi} \right)_k (\nabla \xi)_k = \frac{\partial \mathbf{a}}{\partial \xi} \cdot \nabla \xi \quad (\text{XI.57})$$

Assuming in formulas (XI.56) and (XI.57) that $\xi = r$ (where r is the magnitude of the position vector), we find that

$$[\nabla, \mathbf{a}(r)] = \left[\nabla r, \frac{\partial \mathbf{a}}{\partial r} \right] = \left[\mathbf{e}_r, \frac{\partial \mathbf{a}}{\partial r} \right] \quad (\text{XI.58})$$

$$\nabla \mathbf{a}(r) = \frac{\partial \mathbf{a}}{\partial r} \nabla r = \frac{\partial \mathbf{a}}{\partial r} \frac{\mathbf{r}}{r} \quad (\text{XI.59})$$

[see (XI.48)].

We shall prove the formula

$$\int_V [\nabla \mathbf{a}] dV = \oint_f [\mathbf{n} \mathbf{a}] df = \oint_f [d\mathbf{f}, \mathbf{a}] \quad (\text{XI.60})$$

where \mathbf{a} is a vector function, V is an arbitrary volume, f is the surface confining it, and \mathbf{n} is an outward normal to the surface element df . For this purpose, we find the scalar product of the vector $[\nabla \mathbf{a}]$ and an arbitrary constant vector \mathbf{b} . Inspection of the formula $\nabla [\mathbf{ab}] = \mathbf{b} [\nabla \mathbf{a}] - \mathbf{a} [\nabla \mathbf{b}]$ [see (XI.28)] reveals that $\mathbf{b} [\nabla \mathbf{a}] = \nabla [\mathbf{ab}] + \mathbf{a} [\nabla \mathbf{b}] = \nabla [\mathbf{ab}]$ ($[\nabla \mathbf{b}] = 0$ because $\mathbf{b} = \text{const}$). Now let us integrate the relation obtained over a certain volume V and apply the Ostrogradsky-Gauss theorem to the right-hand side:

$$\int_V \mathbf{b} [\nabla \mathbf{a}] dV = \int_V \nabla [\mathbf{ab}] dV = \oint_f [\mathbf{ab}] \mathbf{n} df$$

Performing a cyclic transposition of the vectors being multiplied in the last integral [see (VI.3)], we obtain $\oint_f \mathbf{b} [\nabla \mathbf{a}] dV = \oint_f \mathbf{b} [\mathbf{na}] df$.

We put the constant vector \mathbf{b} outside the integral:

$$\mathbf{b} \int_V [\nabla \mathbf{a}] dV = \mathbf{b} \oint_f [\mathbf{na}] df$$

This relation must be observed with an arbitrary choice of the vector \mathbf{b} . We can therefore cancel \mathbf{b} . As a result, we arrive at formula (XI.60).

Integral Determination of the Operator ∇ . Let us consider the integral

$$\oint \varphi d\mathbf{f} \quad (\text{XI. 61})$$

over the closed surface f , where φ is an arbitrary scalar function, $d\mathbf{f} = \mathbf{n} df$ is an elementary vector of the area df . Let us take the surface of an infinitely small rectangular parallelepiped with the sides dx , dy , dz as the surface over which the integral is being evaluated. The integral (XI.61) can therefore be written as the sum of six integrals each of which is evaluated over one of the faces of the parallelepiped. Owing to the smallness of the faces, the value of φ within the limits of each face may be considered as constant. Consequently,

$$\begin{aligned} \oint \varphi d\mathbf{f} &= \mathbf{e}_x [\varphi(x+dx, y, z) - \varphi(x, y, z)] dy dz \\ &+ \mathbf{e}_y [\varphi(x, y+dy, z) - \varphi(x, y, z)] dx dz \\ &+ \mathbf{e}_z [\varphi(x, y, z+dz) - \varphi(x, y, z)] dx dy \\ &= \left(\mathbf{e}_x \frac{\partial \varphi}{\partial x} + \mathbf{e}_y \frac{\partial \varphi}{\partial y} + \mathbf{e}_z \frac{\partial \varphi}{\partial z} \right) dx dy dz = \nabla \varphi dV \end{aligned}$$

whence

$$\nabla \varphi = \lim_{V \rightarrow 0} \frac{1}{V} \oint \varphi d\mathbf{f} \quad (\text{XI.62})$$

In accordance with formula (XI.62), the operator ∇ can be determined as follows:

$$\nabla = \lim_{V \rightarrow 0} \frac{1}{V} \oint df \quad (\text{XI.63})$$

Let us take a volume V and divide it into elementary volumes ΔV . By formula (XI.62), the approximate equality $\nabla \varphi \cdot \Delta V \approx \oint \varphi df$ is observed for each volume.

Now let us summate this equation over all the elementary volumes. In summation, the right-hand sides corresponding to the boundaries between adjacent volumes mutually eliminate one another. Only the terms corresponding to the outer surface remain uncompensated. Therefore in passing to the limit at which all the ΔV 's shrink to zero, we arrive at the relation

$$\int_V \nabla \varphi dV = \int_f \varphi df \quad (\text{XI.64})$$

According to (XI.64), the integral of the scalar function φ over the closed surface f can be transformed into an integral over the volume confined in it by replacing the surface element df with the operator ∇dV . Here the components df experience the transformation

$$df_i \rightarrow dV \frac{\partial}{\partial x_i} \quad (\text{XI.65})$$

Naturally, the reverse transformation is also possible.

Matters are similar with the integral of a vector function. Indeed, according to the Ostrogradsky-Gauss theorem [see (XI.13)], we have $\int_V \nabla a dV = \oint_f a df$, which also agrees with (XI.65).

In general, the transformation (XI.65) is always allowable regardless of the specific kind of expression in the integrands. An example of such a transformation is relation (XI.60).

Curvilinear Coordinates. It is sometimes convenient to determine the position of a point not by means of the Cartesian coordinates x_1, x_2, x_3 , but by using three other numbers q_1, q_2, q_3 corresponding better to the nature of the problem being considered. These numbers are known as the curvilinear coordinates of a point.

Imposing (if the need appears) restrictions on the region of changing of curvilinear coordinates, we can achieve a one-to-one correspondence of the variables x_i and q_i . Hence, $q_i = q_i(x_1, x_2, x_3)$, and $x_i = x_i(q_1, q_2, q_3)$ ($i = 1, 2, 3$).

Surfaces described by the equation $q_i(x_1, x_2, x_3) = \text{const}$ are said to be **coordinate surfaces**. The lines of intersection of two coordinate surfaces are called **coordinate lines**. Only one coordinate varies along a coordinate line; the other two remain unchanged.

An example of curvilinear coordinates is the spherical system r, ϑ, φ . In this case, (1) spheres, $r = \text{const}$, (2) half-angle cones, $\vartheta = \text{const}$, and (3) half-planes, $\varphi = \text{const}$, are the coordinate surfaces. The coordinate lines are (1) radii—the lines r , (2) meridians—the lines ϑ , and (3) parallels—the lines φ .

If the coordinate lines at every point are mutually perpendicular, the curvilinear coordinates are said to be **orthogonal**. We shall limit ourselves only to a treatment of orthogonal coordinates. They include spherical and cylindrical coordinates.

Let us introduce for every point P the unit vectors \mathbf{e}_1^* , \mathbf{e}_2^* , \mathbf{e}_3^* directed along tangents to the coordinate lines at the given point towards an increase in the relevant variables q_i . Owing to orthogonality, the following relations hold for the unit vectors \mathbf{e}_i^* :

$$\mathbf{e}_i^* \mathbf{e}_k^* = \delta_{ik} \quad (\text{XI.66})$$

We determine the derivative of the position vector $\mathbf{r} = \mathbf{r}(q_1, q_2, q_3)$ with respect to the coordinate q_i . The remaining two coordinates do not change in differentiation. Consequently, when the coordinate q_i is given the increment δq_i , the tip of the vector \mathbf{r} travels along the coordinate line q_i . Therefore, the vector $\partial \mathbf{r} / \partial q_i$ is directed along a tangent to the coordinate line q_i , i.e. is collinear with \mathbf{e}_i^* . Denoting the magnitude of the vector $\partial \mathbf{r} / \partial q_i$ by the symbol H_i , we can write that

$$\frac{\partial \mathbf{r}}{\partial q_i} = H_i \mathbf{e}_i^* \quad (i = 1, 2, 3) \quad (\text{XI.67})$$

Representing \mathbf{r} in the form of $\sum x_k \mathbf{e}_k$ (here \mathbf{e}_k are the unit vectors of the Cartesian coordinate axes), we can write that

$$\frac{\partial \mathbf{r}}{\partial q_i} = \sum_k \frac{\partial x_k}{\partial q_i} \mathbf{e}_k = H_i \mathbf{e}_i^* \quad (i = 1, 2, 3)$$

Squaring of this relation yields

$$H_i^2 = \sum_k \left(\frac{\partial x_k}{\partial q_i} \right)^2 \quad (i = 1, 2, 3) \quad (\text{XI.68})$$

The quantities H_i are known as the **Lamé coefficients**. Their values can be found by formula (XI.68) if we know the form of the functions $x_k = x_k(q_i)$. For example, for a spherical coordinate system, $x_1 = r \sin \vartheta \cos \varphi$, $x_2 = r \sin \vartheta \sin \varphi$, $x_3 = r \cos \vartheta$. Hence

$$\frac{\partial x_1}{\partial q_1} = \frac{\partial x_1}{\partial r} = \sin \vartheta \cos \varphi, \quad \frac{\partial x_2}{\partial q_1} = \frac{\partial x_2}{\partial r} = \sin \vartheta \sin \varphi, \quad \frac{\partial x_3}{\partial q_1} = \frac{\partial x_3}{\partial r} = \cos \vartheta$$

so that by (XI.68)

$$H_1^2 = (\sin \vartheta \cos \varphi)^2 + (\sin \vartheta \sin \varphi)^2 + (\cos \vartheta)^2 = 1$$

Similar calculations for H_2 and H_3 yield

$$H_1 = 1, \quad H_2 = r, \quad H_3 = r \sin \vartheta \quad (\text{XI.69})$$

Let us find the square of the distance ds between two points. By (XI.67), we have

$$dr = \sum_i \frac{\partial \mathbf{r}}{\partial q_i} dq_i = \sum_i H_i \mathbf{e}_i^* dq_i \quad (\text{XI.70})$$

Consequently,

$$ds^2 = |dr|^2 = \sum_i H_i^2 dq_i^2 \quad (\text{XI.71})$$

For a spherical coordinate system, this formula yields

$$ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \quad (\text{XI.72})$$

Let us draw coordinate surfaces through the tail and the tip of the vector dr . The result is an infinitely small parallelepiped whose edges, by (XI.70), are

$$dl_i = H_i dq_i \quad (i = 1, 2, 3) \quad (\text{XI.73})$$

The faces of this parallelepiped have the areas

$$df_1 = H_2 H_3 dq_2 dq_3, \quad df_2 = H_3 H_1 dq_3 dq_1, \quad df_3 = H_1 H_2 dq_1 dq_2 \quad (\text{XI.74})$$

The volume of the parallelepiped is

$$dV = H_1 H_2 H_3 dq_1 dq_2 dq_3 \quad (\text{XI.75})$$

Instead of calculation by formulas (XI.68), the Lamé coefficients can be found with the aid of expressions (XI.73). For instance, for spherical coordinates (Fig. XI.6), an elementary parallelepiped has the edges $dl_1 = dr$, $dl_2 = r d\vartheta$, and $dl_3 = r \sin \vartheta d\varphi$, whence we directly obtain the values (XI.69) for the Lamé coefficients.

For a cylindrical coordinate system (Fig. XI.7), the edges of an elementary parallelepiped are $dl_1 = d\rho$, $dl_2 = \rho d\varphi$, and $dl_3 = dz$, whence

$$H_1 = 1, \quad H_2 = \rho, \quad H_3 = 1 \quad (\text{XI.76})$$

Let us find the expression for the gradient in curvilinear coordinates. By formula (XI.5), the projection of the gradient of the function ψ onto the direction of the unit vector \mathbf{e}_i^* is $\partial\psi/\partial l_i$. With a view to (XI.73), we obtain

$$\frac{\partial\psi}{\partial l_i} = \frac{1}{H_i} \frac{\partial\psi}{\partial q_i}$$

The gradient itself is determined by the formula

$$\nabla\psi = \sum_i \frac{\partial\psi}{\partial l_i} \mathbf{e}_i^* = \sum_i \frac{1}{H_i} \frac{\partial\psi}{\partial q_i} \mathbf{e}_i^* \quad (\text{XI.77})$$

Consequently, in a spherical coordinate system

$$\nabla\psi = \frac{\partial\psi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial\psi}{\partial\vartheta} \mathbf{e}_\vartheta + \frac{1}{r \sin\vartheta} \frac{\partial\psi}{\partial\varphi} \mathbf{e}_\varphi \quad (\text{XI.78})$$

and in a cylindrical system

$$\nabla\psi = \frac{\partial\psi}{\partial\rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial\psi}{\partial\varphi} \mathbf{e}_\varphi + \frac{\partial\psi}{\partial z} \mathbf{e}_z \quad (\text{XI.79})$$

To determine the divergence in curvilinear coordinates, we shall proceed from the definition (XI.7) according to which

$$\nabla\mathbf{a} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_V \mathbf{a} d\mathbf{f} = \lim_{V \rightarrow 0} \frac{\Phi}{V} \quad (\text{XI.80})$$

Let us take as V the volume of an infinitely small parallelepiped including the point P for which the divergence is being calculated.

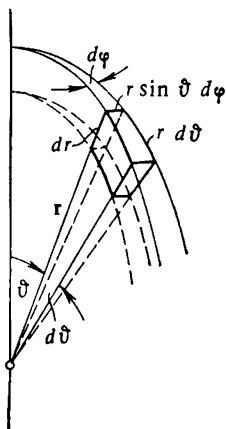


Fig. XI.6.

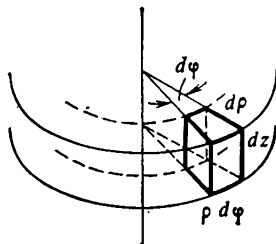


Fig. XI.7.

Reasoning in the same way as when obtaining formula (XI.11), let us write an expression for the flux through the faces perpendicular to the unit vector \mathbf{e}_1^* :

$\Phi_1 = a_1' \Delta f_1' - a_1'' \Delta f_1'' = (a_1 H_2 H_3)' \Delta q_2 \Delta q_3 - (a_1 H_2 H_3)'' \Delta q_2 \Delta q_3$ [see (XI.74)]. Here a_1 is the projection of the vector \mathbf{a} onto the direction of \mathbf{e}_1^* . One prime indicates the values of quantities relating to one of two opposite faces, and two primes indicate the values relating to the other face.

Expanding $a_1 H_2 H_3$ into a series in the vicinity of the point P , we obtain

$$\Phi_1 = \frac{\partial(a_1 H_2 H_3)}{\partial q_1} \Delta q_1 \Delta q_2 \Delta q_3 + \varepsilon_1 = \frac{1}{H_1 H_2 H_3} \frac{\partial(a_1 H_2 H_3)}{\partial q_1} V + \varepsilon_1$$

where V is the volume of the parallelepiped [see (XI.75)], and ε_1 is a quantity of a higher order of smallness than V . The fluxes through two other pairs of faces are evaluated in a similar way:

$$\Phi_2 = \frac{1}{H_1 H_2 H_3} \frac{\partial (a_2 H_3 H_1)}{\partial q_2} V + \varepsilon_2, \quad \Phi_3 = \frac{1}{H_1 H_2 H_3} \frac{\partial (a_3 H_1 H_2)}{\partial q_3} V + \varepsilon_3$$

Substituting $\Phi = \Phi_1 + \Phi_2 + \Phi_3$ into (XI.80) and performing a limiting process, we arrive at the formula

$$\nabla \mathbf{a} = \frac{1}{H_1 H_2 H_3} \left\{ \frac{\partial (a_1 H_2 H_3)}{\partial q_1} + \frac{\partial (a_2 H_3 H_1)}{\partial q_2} + \frac{\partial (a_3 H_1 H_2)}{\partial q_3} \right\} \quad (\text{XI.81})$$

Taking (XI.69) into account, we obtain an expression for the divergence in spherical coordinates:

$$\nabla \mathbf{a} = \frac{1}{r^2} \frac{\partial (r^2 a_r)}{\partial r} + \frac{1}{r \sin \vartheta} \frac{\partial (\sin \vartheta a_\vartheta)}{\partial \vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial a_\varphi}{\partial \varphi} \quad (\text{XI.82})$$

In cylindrical coordinates

$$\nabla \mathbf{a} = \frac{1}{\rho} \frac{\partial (\rho a_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z} \quad (\text{XI.83})$$

By (XI.16), the projection of a curl onto the direction of the unit vector \mathbf{e}_i^* is determined by the expression

$$[\nabla \mathbf{a}]_i = \lim_{f_i \rightarrow 0} \frac{1}{f_i} \oint \mathbf{a} d\mathbf{l} = \lim_{f_i \rightarrow 0} \frac{C_i}{f_i} \quad (\text{XI.84})$$

where f_i is a surface element perpendicular to \mathbf{e}_i^* , and C_i is the circulation of the vector \mathbf{a} around the contour confining this element.

To evaluate the projection of $[\nabla \mathbf{a}]$ onto the unit vector \mathbf{e}_1^* , let us take f_1 as an infinitely small curvilinear rectangle with the sides $\Delta l_2 = H_2 \Delta q_2$ and $\Delta l_3 = H_3 \Delta q_3$. The circulation of \mathbf{a} around the contour of this rectangle can be represented in the form of four integrals like $\int \mathbf{a} d\mathbf{l}$. Two of them are evaluated over the opposite sides $\Delta l'_2$ and $\Delta l''_2$, and two over the opposite sides $\Delta l'_3$ and $\Delta l''_3$ (Fig. XI.8). As a result, with a view to (XI.73), we obtain

$$\begin{aligned} C_1 &= a'_2 \Delta l'_2 + a''_3 \Delta l''_3 - a''_2 \Delta l''_2 - a'_3 \Delta l'_3 \\ &= (a''_3 \Delta l''_3 - a'_3 \Delta l'_3) - (a''_2 \Delta l''_2 - a'_2 \Delta l'_2) \\ &= (a''_3 H''_3 - a'_3 H'_3) \Delta q_3 - (a''_2 H''_2 - a'_2 H'_2) \Delta q_2 \quad (\text{XI.85}) \end{aligned}$$

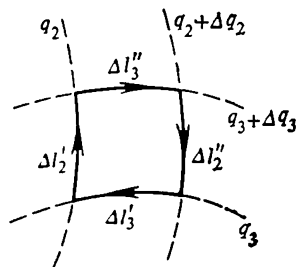


Fig. XI.8.

Expanding $a_3 H_3$ and $a_2 H_2$ into a series in the vicinity of the point P for which the curl is being calculated, we can write that

$$a_3'' H_3'' - a_3' H_3' = \frac{\partial (a_3 H_3)}{\partial q_2} \Delta q_2 + \varepsilon_3, \quad a_2'' H_2'' - a_2' H_2' = \frac{\partial (a_2 H_2)}{\partial q_3} \Delta q_3 + \varepsilon_2$$

Consequently, expression (XI.85) can be written as

$$\begin{aligned} C_1 &= \frac{\partial (a_3 H_3)}{\partial q_2} \Delta q_2 \Delta q_3 - \frac{\partial (a_2 H_2)}{\partial q_3} \Delta q_3 \Delta q_2 + \varepsilon \\ &= \frac{1}{H_2 H_3} \left\{ \frac{\partial (a_3 H_3)}{\partial q_2} - \frac{\partial (a_2 H_2)}{\partial q_3} \right\} f_1 + \varepsilon \end{aligned}$$

where f_1 is the area of the rectangle [see (XI.74)] and ε is a quantity of a higher order of smallness than f_1 .

Introducing the value of C_1 we have found into (XI.84) and performing a limiting process, we arrive at the formula

$$[\nabla \mathbf{a}]_1 = \frac{1}{H_2 H_3} \left\{ \frac{\partial (a_3 H_3)}{\partial q_2} - \frac{\partial (a_2 H_2)}{\partial q_3} \right\}$$

Similar formulas are obtained for the two other projections of the vector $[\nabla \mathbf{a}]$. All three formulas can be combined into a single one:

$$[\nabla \mathbf{a}]_i = \frac{1}{H_k H_l} \left\{ \frac{\partial (a_l H_l)}{\partial q_k} - \frac{\partial (a_k H_k)}{\partial q_l} \right\} \quad (i = 1, 2, 3) \quad (\text{XI.86})$$

(the subscripts i, k, l form a cyclic transposition of the sequence 1, 2, 3). Finally, let us find an expression for the Laplacian operator in curvilinear coordinates. By (XI.41), we have $\Delta \psi = \nabla (\nabla \psi) = \text{div grad } \psi$.

Using formulas (XI.81) and (XI.77), we obtain

$$\begin{aligned} \Delta \psi &= \nabla (\nabla \psi) = \frac{1}{H_1 H_2 H_3} \left\{ \frac{\partial}{\partial q_1} \left(\frac{H_2 H_3}{H_1} \frac{\partial \psi}{\partial q_1} \right) \right. \\ &\quad \left. + \frac{\partial}{\partial q_2} \left(\frac{H_3 H_1}{H_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{H_1 H_2}{H_3} \frac{\partial \psi}{\partial q_3} \right) \right\} \quad (\text{XI.87}) \end{aligned}$$

Introducing the values (XI.69), we obtain an expression for the Laplacian operator in spherical coordinates

$$\begin{aligned} \Delta \psi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial \psi}{\partial \vartheta} \right) \\ &\quad + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 \psi}{\partial \varphi^2} \quad (\text{XI.88}) \end{aligned}$$

In cylindrical coordinates

$$\Delta \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2} \quad (\text{XI.89})$$

XII. Four-Dimensional Vectors and Tensors in Pseudo-Euclidean Space

In Appendices VI and X, we considered vectors and tensors in Euclidean space in which the square of the position vector is determined by the expression

$$x_1^2 + x_2^2 + \dots + x_n^2 = \sum_{i=1}^n x_i^2 \quad (\text{XII.1})$$

If we introduce the tensor

$$(g_{ih}) = (\delta_{ih}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} \quad (\text{XII.2})$$

expression (XII.1) can be written as

$$\sum_{i,h=1}^n g_{ih} x_i x_h \quad (\text{XII.3})$$

The tensor g_{ih} is called a **metric tensor**.

The equations of the special theory of relativity and electrodynamics acquire an especially simple and clear nature if they are written as relations between vectors and tensors in four-dimensional space whose metric is determined by the tensor

$$(g^{\mu\nu}) = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{XII.4})$$

For reasons that will be revealed later on, we shall use the following notation. We shall write the indices on the components of a four-position vector not as subscripts, but as superscripts. We shall write the Kronecker symbol as δ_{ν}^{μ} , i.e. with one subscript and one superscript. We shall use superscripts or subscripts on the components of the metric tensor depending on the circumstances.

We shall adhere to the following rules in writing formulas:

1. In each pair of dummy indices (i.e. indices over which summation is to be performed), one will be a superscript and the other a subscript.

2. Free indices (i.e. those over which summation is not performed) will be placed in the same position—either as a superscript or a subscript—on both sides of an equation.

In the coefficients of the linear transformation of vector and tensor components, we shall also make one index a superscript and the

other a subscript¹. Consequently, unlike the transformation formulas $a'_i = \sum_k \alpha_{ik} a_k$ which we had to do with in Appendices VI and X, we shall write similar formulas as follows: $a'^\mu = \sum_\nu \alpha^\mu_\nu a^\nu$.

Recall that in accordance with the condition we have adopted, whenever the symbol \sum is absent, summation even over paired indices is not performed.

A space whose properties are determined by the tensor (XII.4) is called a **pseudo-Euclidean** one. We shall distinguish the coordinates, and also vector and tensor components in this four-space with the aid of the Greek indices μ, ν , etc., that can take on the four values 0, 1, 2, and 3. We shall use the Roman indices i, k, \dots (running through the values 1, 2, and 3) on coordinates and vector components in conventional three-dimensional space (in space with a Euclidean metric). We shall sometimes use Roman indices on the components of four-vectors and four-tensors to underline that we have in mind non-zero values of the indices.

Having determined the square of a position vector in the four-space being considered by means of an expression similar to (XII.3) and taking into account (XII.4), we obtain

$$\sum_{\mu, \nu=0}^3 g_{\mu\nu} x^\mu x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \quad (\text{XII.5})$$

In the theory of relativity, the product of the time t and the speed of light in a vacuum is taken as the coordinate x^0 , and the coordinates in conventional three-dimensional space are taken as the remaining coordinates x^i . Hence,

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z \quad (\text{XII.6})$$

Having this in view, we can write expression (XII.5) as

$$\sum_{\mu, \nu=0}^3 g_{\mu\nu} x^\mu x^\nu = (x^0)^2 - \sum_{i=1}^3 (x^i)^2 = c^2 t^2 - \mathbf{r}^2 \quad (\text{XII.7})$$

Formulas for the Transformation of Coordinates. In a transition to another coordinate system, the components of a four-position vector are transformed according to the linear law:

$$x'^\mu = \sum_{\nu=0}^3 \alpha^\mu_\nu x^\nu \quad (\text{XII.8})$$

¹ We shall see in the following that one index is a superscript and the other one is a subscript on mixed components of tensors. It must be remembered, however, that linear transformation coefficients do not have the properties of tensor components.

The inverse transformation is performed by the formula

$$x^\mu = \sum_{\nu=0}^3 \bar{\alpha}_\nu^\mu x'^\nu \quad (\text{XII.9})$$

where $\bar{\alpha}_\nu^\mu$ are the coefficients of the inverse transformation.

Owing to the invariance of the square of the four-position vector, the condition must be observed that $\sum_{\mu, \nu=0}^3 g_{\mu\nu} x^\mu x^\nu = \sum_{\mu, \nu=0}^3 g_{\mu\nu} x'^\mu x'^\nu$. Substituting for x'^μ and x'^ν in this expression their values from (XII.8), we obtain

$$\sum_{\mu, \nu=0}^3 g_{\mu\nu} x^\mu x^\nu = \sum_{\mu, \nu=0}^3 g_{\mu\nu} \sum_{\rho=0}^3 \alpha_\rho^\mu x^\rho \sum_{\sigma=0}^3 \alpha_\sigma^\nu x^\sigma$$

Let us exchange on the right the places of the dummy indices μ and ρ , and also of ν and σ . The result is $\sum_{\mu, \nu=0}^3 g_{\mu\nu} x^\mu x^\nu = \sum_{\mu, \nu=0}^3 x^\mu x^\nu \sum_{\rho, \sigma=0}^3 g_{\rho\sigma} \alpha_\mu^\rho \alpha_\nu^\sigma$.

This gives us the conditions which the coefficients of the linear transformation (XII.8) must satisfy:

$$\sum_{\rho, \sigma=0}^3 g_{\rho\sigma} \alpha_\mu^\rho \alpha_\nu^\sigma = g_{\mu\nu}$$

Having in view that $g_{\rho\sigma}$ differ from zero only when $\rho = \sigma$ [see (XII.4)], this relation can be simplified as follows:

$$\sum_{\rho=0}^3 g_{\rho\rho} \alpha_\mu^\rho \alpha_\nu^\rho = g_{\mu\nu} \quad (\mu, \nu = 0, 1, 2, 3) \quad (\text{XII.10})$$

For example, for $\mu = \nu = 0$, we obtain by this formula

$$(\alpha_0^0)^2 - (\alpha_0^1)^2 - (\alpha_0^2)^2 - (\alpha_0^3)^2 = 1 \quad (\text{XII.11})$$

and for $\mu = 1$ and $\nu = 2$:

$$\alpha_1^0 \alpha_2^0 - \alpha_1^1 \alpha_2^1 - \alpha_1^2 \alpha_2^2 - \alpha_1^3 \alpha_2^3 = 0 \quad (\text{XII.12})$$

We must note that if we took the tensor (XII.2) as $g_{\rho\sigma}$, we could reduce formula (XII.10) to the form $\sum_{\rho} \alpha_\mu^\rho \alpha_\nu^\rho = \delta_{\mu\nu}$, which coincides with formula (VI.40).

It is obvious that relations similar to (XII.10) also hold for the coefficients $\bar{\alpha}_\nu^\mu$.

Now let us find the relation between the coefficients of the direct and inverse transformations. Remember that in Euclidean space, this relation is $\bar{\alpha}_{ik} = \alpha_{ki}$ [compare formulas (VI.37) and (VI.38)].

Let us consecutively apply the transformations (XII.9) and (XII.8). The result is

$$x^\mu = \sum_{\rho=0}^3 \bar{\alpha}_\rho^\mu x'^\rho = \sum_{\rho=0}^3 \bar{\alpha}_\rho^\mu \sum_{\nu=0}^3 \alpha_\nu^\rho x^\nu = \sum_{\nu=0}^3 x^\nu \sum_{\rho=0}^3 \bar{\alpha}_\rho^\mu \alpha_\nu^\rho$$

The component x^μ can be written as $\sum_\nu \delta_\nu^\mu x^\nu$. It thus follows that

$$\sum_{\rho=0}^3 \bar{\alpha}_\rho^\mu \alpha_\nu^\rho = \delta_\nu^\mu \quad (\mu, \nu = 0, 1, 2, 3) \quad (\text{XII.13})$$

It is not difficult to see that the system of equations (XII.13) will be satisfied if we assume that

$$\bar{\alpha}_0^0 = \alpha_0^0, \quad \bar{\alpha}_i^0 = -\alpha_0^i, \quad \bar{\alpha}_0^i = -\alpha_i^0, \quad \bar{\alpha}_i^k = \alpha_i^k \quad (i, k = 1, 2, 3) \quad (\text{XII.14})$$

Indeed, let $\mu = \nu = 0$. Equation (XII.13) therefore becomes $\sum_{\rho=0}^3 \bar{\alpha}_\rho^0 \alpha_0^\rho = 1$ or, with account of (XII.14), $\alpha_0^0 \alpha_0^0 - \sum_{i=1}^3 \alpha_0^i \alpha_0^i = 1$ which agrees with (XII.11).

Assume that $\mu = 1$ and $\nu = 2$. Equation (XII.13) now becomes $\sum_{\rho=0}^3 \bar{\alpha}_\rho^1 \alpha_2^\rho = 0$ or, with account of (XII.14), $-\alpha_1^0 \alpha_2^0 + \sum_{i=1}^3 \alpha_1^i \alpha_2^i = 0$ which agrees with (XII.12). We can convince ourselves similarly that the remaining 14 equations contained in (XII.13) are satisfied by the solution (XII.14).

We must note that relations (XII.14) can be written as a single relation

$$\bar{\alpha}_\nu^\mu g_{\mu\mu} = \alpha_\mu^\nu g_{\nu\nu} \quad (\text{XII.15})$$

Indeed, when $\mu = \nu = 0$, we have $g_{\mu\mu} = g_{\nu\nu}$, therefore $\bar{\alpha}_0^0 = \alpha_0^0$; when $\mu = 0$ and $\nu = i \neq 0$, as well as when $\mu = i \neq 0$ and $\nu = 0$, we have $g_{\mu\mu} = -g_{\nu\nu}$, owing to which $\bar{\alpha}_i^0 = -\alpha_0^i$ and $\bar{\alpha}_0^i = -\alpha_i^0$; finally, when $\mu = i \neq 0$ and $\nu = k \neq 0$, both factors ($g_{\mu\mu}$ and $g_{\nu\nu}$) equal -1 , so that $\bar{\alpha}_k^i = \alpha_i^k$. As a result, we have arrived at relations (XII.14).

The theory of relativity usually deals with transformations in which the coordinates $x^2 = y$ and $x^3 = z$ remain unchanged ($x'^2 = x^2$ and $x'^3 = x^3$). The matrix of the transformation coefficients in this case can evidently be written as follows:

$$[\alpha_\nu^\mu] = \begin{bmatrix} \alpha_0^0 & \alpha_1^0 & 0 & 0 \\ \alpha_0^1 & \alpha_1^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (\text{XII.16})$$

The matrix of the inverse transformation coefficients has a similar appearance.

Let us designate by p a parameter that can be used to characterize the difference between the systems K and K' . It can be, for example, the angle of rotation of one system relative to the other. In the theory of relativity, the role of the parameter p is played by the relative velocity of the systems K and K' . It is evident that as a result of a limiting process in which p shrinks to zero, both systems coincide, so that the matrix (XII.16) is

$$[\alpha_{\nu}^{\mu}]_{p \rightarrow 0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$\lim_{p \rightarrow 0} \alpha_0^0 = \lim_{p \rightarrow 0} \alpha_1^1 = 1, \quad \lim_{p \rightarrow 0} \alpha_1^0 = \lim_{p \rightarrow 0} \alpha_0^1 = 0 \quad (\text{XII.17})$$

The four coefficients α_{ν}^{μ} differing from zero and unity are not independent. Having written relation (XII.10) for $\mu = \nu = 0$, we find that

$$(\alpha_0^0)^2 - (\alpha_0^1)^2 = 1 \quad (\text{XII.18})$$

A similar relation also holds for the coefficients of the inverse transformation: $(\bar{\alpha}_0^0)^2 - (\bar{\alpha}_0^1)^2 = 1$, whence with a view to (XII.14) we obtain the equation

$$(\alpha_0^0)^2 - (\alpha_1^0)^2 = 1 \quad (\text{XII.19})$$

Now let us write condition (XII.10) for $\mu = 0$ and $\nu = 1$:

$$\alpha_0^0 \alpha_1^0 - \alpha_0^1 \alpha_1^1 = 0 \quad (\text{XII.20})$$

A comparison of Eqs. (XII.18) and (XII.19) shows that $(\alpha_0^1)^2 = (\alpha_1^0)^2$, i.e. $\alpha_0^1 = \pm \alpha_1^0$. A comparison of this condition with Eq. (XII.20) gives two possible alternatives of the relations between the coefficients:

$$(1) \alpha_0^0 = \alpha_1^1 \quad \text{if} \quad \alpha_0^1 = \alpha_1^0, \quad \text{and} \quad (2) \alpha_0^0 = -\alpha_1^1 \quad \text{if} \quad \alpha_0^1 = -\alpha_1^0$$

The first alternative agrees with the property of the coefficients expressed by formula (XII.17). Consequently, we must assume that $\alpha_0^0 = \alpha_1^1$, and $\alpha_1^0 = \alpha_0^1$. Introducing the notation $\alpha_0^0 = \alpha_1^1 = \alpha_0$ and $\alpha_1^0 = \alpha_0^1 = \alpha_1$, we can write the matrix of the transformation coefficients as follows:

$$[\alpha_{\nu}^{\mu}] = \begin{bmatrix} \alpha_0 & \alpha_1 & 0 & 0 \\ \alpha_1 & \alpha_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (\text{XII.21})$$

This matrix contains only one independent coefficient (whose value is determined by the specific form of the transformation) because $\alpha_0 = \alpha_0^0$ and $\alpha_1 = \alpha_1^0$ are related by Eq. (XII.19):

$$\alpha_0^2 - \alpha_1^2 = 1 \quad (\text{XII.22})$$

By (XII.14), $\bar{\alpha}_0^0 = \alpha_0^0 = \alpha_0$, and $\bar{\alpha}_1^0 = -\alpha_0^1 = -\alpha_1$. Therefore, the matrix of the inverse transformation is

$$[\bar{\alpha}_\nu^\mu] = \begin{bmatrix} \alpha_0 & -\alpha_1 & 0 & 0 \\ -\alpha_1 & \alpha_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (\text{XII.23})$$

Contravariant and Covariant Vectors. Substitution into (XII.8) of the values of α_ν^μ from (XII.21) leads to formulas for the transformation of the components of a four-position vector:

$$x'^0 = \alpha_0 x^0 + \alpha_1 x^1, \quad x'^1 = \alpha_1 x^0 + \alpha_0 x^1, \quad x'^2 = x^2, \quad x'^3 = x^3 \quad (\text{XII.24})$$

The set of the four quantities a^0, a^1, a^2 , and a^3 that transform in a transition from one coordinate system to another according to the same rules as the components of a four-position vector, i.e. according to the formulas

$$a'^\mu = \sum_{\nu=0}^3 \alpha_\nu^\mu a^\nu \quad (\text{XII.25})$$

is said to be a **four-dimensional vector** (a four-vector). The coefficients α_ν^μ in (XII.25) have the same values as in (XII.8).

The component a^0 corresponds to the component x^0 of a four-position vector that in the theory of relativity is taken equal to ct [see (XII.6)]. In this connection, the component a^0 is called a time one. The components a^1, a^2, a^3 correspond to the components $x^1 = x, x^2 = y, x^3 = z$, and are therefore called spatial components.

Substitution into (XII.25) of the values of α_ν^μ from (XII.21) leads to the following formulas:

$$a'^0 = \alpha_0 a^0 + \alpha_1 a^1, \quad a'^1 = \alpha_1 a^0 + \alpha_0 a^1, \quad a'^2 = a^2, \quad a'^3 = a^3 \quad (\text{XII.26})$$

The square of a four-vector is determined by analogy with the square of a four-position vector [see (XII.5)]:

$$\sum_{\mu=0}^3 g_{\mu\mu} a^\mu a^\mu = (a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2 \quad (\text{XII.27})$$

We must note that the square of a four-vector can be either positive or negative; particularly, it may be zero.

In writing vector formulas for a pseudo-Euclidean space, we become confronted with a major inconvenience because the square of a four-vector is determined by expression (XII.27) that cannot be written

in a compact form as $\sum (a^\mu)^2$. This inconvenience is eliminated by introducing two kinds of four-vectors. They are distinguished by using a superscript on vectors of one kind and a subscript on vectors of the other kind. The first kind of vectors are called **contravariant**, and the second, **covariant**. Hence, by using a superscript on the components of a four-position vector, we have related this vector to the category of contravariant ones.

A covariant vector a_μ corresponds to each contravariant vector a^μ (and vice versa), and it is assumed that

$$a_\mu = g_{\mu\alpha} a^\alpha \quad (\text{XII.28})$$

[see (XII.4)], i.e.

$$a_0 = a^0, \quad a_1 = -a^1, \quad a_2 = -a^2, \quad a_3 = -a^3 \quad (\text{XII.29})$$

It is not difficult to see that relation (XII.28) can be written as

$$a^\mu = g^{\mu\alpha} a_\alpha \quad (\text{XII.30})$$

Consequently, raising or lowering of the index μ on the component of a four-vector is attended by multiplication of the component by $g_{\mu\alpha}$ or $g^{\mu\alpha}$.

When using contravariant and covariant components, the square of a four-vector [see (XII.27)] can be written as follows:

$$\sum_{\mu=0}^3 a^\mu a_\mu = a^0 a_0 + a^1 a_1 + a^2 a_2 + a^3 a_3 \quad (\text{XII.31})$$

By analogy with formula (XII.31), which gives the product of two identical four-vectors, we can determine the scalar product of two different four-vectors:

$$\sum_{\mu=0}^3 a^\mu b_\mu = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3 = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \quad (\text{XII.32})$$

The following expression is obvious:

$$\sum_{\mu=0}^3 a^\mu b_\mu = \sum_{\mu=0}^3 a_\mu b^\mu \quad (\text{XII.33})$$

In general in any pair of dummy indices, we may exchange the places of the superscript and subscript.

It must be noted that in purely spatial rotations (i.e. transformations not affecting the component a^0), the three spatial components of the four-vector a^μ behave like the components of a vector in three-dimensional Euclidean space (the component a^0 behaves here like a three-dimensional scalar). In this connection, the components of a four-vector can be written as

$$a^\mu = (a^0, a_i) \quad (i = 1, 2, 3) \quad (\text{XII.34})$$

or, more briefly,

$$a^\mu = (a^0, \mathbf{a}) \quad (\text{XII.35})$$

The covariant components of the same vector are as follows:

$$a_\mu = (a^0, -a_i) \quad (i = 1, 2, 3) \quad (\text{XII.36})$$

or

$$a_\mu = (a^0, -\mathbf{a}) \quad (\text{XII.37})$$

The square of a four-vector can be expressed as

$$\sum_\mu a^\mu a_\mu = (a^0)^2 - \sum_i a_i^2 = (a^0)^2 - \mathbf{a}^2 \quad (\text{XII.38})$$

and the scalar product of four-vectors as

$$\sum_\mu a^\mu b_\mu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b} \quad (\text{XII.39})$$

It is a simple matter to see that the increment of the square of a four-vector can be written in two ways:

$$\delta \sum_\mu a^\mu a_\mu = 2 \sum_\mu a^\mu \delta a_\mu = 2 \sum_\mu a_\mu \delta a^\mu \quad (\text{XII.40})$$

Four-Dimensional Gradient. Assume that we are given a scalar function of the quantities x^μ :

$$\varphi(x^0, x^1, x^2, x^3)$$

According to the rules of differential calculus, the increment of this function is given by the expression

$$d\varphi = \frac{\partial \varphi}{\partial x^0} dx^0 + \frac{\partial \varphi}{\partial x^1} dx^1 + \frac{\partial \varphi}{\partial x^2} dx^2 + \frac{\partial \varphi}{\partial x^3} dx^3$$

The increment of a function cannot depend on the coordinate system it is being calculated in, i.e. is an invariant. We thus conclude that the quantities

$$\frac{\partial \varphi}{\partial x^0}, \frac{\partial \varphi}{\partial x^1}, \frac{\partial \varphi}{\partial x^2}, \frac{\partial \varphi}{\partial x^3} \quad (\text{XII.41})$$

behave like covariant components of a vector in transformations of the coordinates. This vector is called the **four-gradient** of the function φ .

If we want to introduce the four-operator of the gradient (the operator ∇^*), its covariant components must be determined as follows:

$$\nabla_0^* = \frac{\partial}{\partial x^0}, \quad \nabla_1^* = \frac{\partial}{\partial x^1}, \quad \nabla_2^* = \frac{\partial}{\partial x^2}, \quad \nabla_3^* = \frac{\partial}{\partial x^3} \quad (\text{XII.42})$$

or

$$\nabla_\mu^* = \left(\frac{\partial}{\partial x^0}, \nabla \right) \quad (\text{XII.43})$$

where ∇ is the operator of the gradient in three-dimensional Euclidean space. Consequently, the contravariant components of the operator ∇^* will be

$$\nabla^{*0} = \frac{\partial}{\partial x^0}, \quad \nabla^{*1} = -\frac{\partial}{\partial x^1}, \quad \nabla^{*2} = -\frac{\partial}{\partial x^2}, \quad \nabla^{*3} = -\frac{\partial}{\partial x^3} \quad (\text{XII.44})$$

which can be written as follows:

$$\nabla^{*\mu} = \left(\frac{\partial}{\partial x^0}, -\nabla \right) \quad (\text{XII.45})$$

We are now able to explain the essence of the difference between contravariant and covariant vectors. By (XII.8), the derivative of x'^μ with respect to x^ν is α_ν^μ :

$$\alpha_\nu^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \quad (\text{XII.46})$$

Introducing these values of α_ν^μ into (XII.8), we can give the formula for transforming the contravariant components of a four-vector to the form

$$a'^\mu = \sum_{\nu=0}^3 \frac{\partial x'^\mu}{\partial x^\nu} a^\nu \quad (\text{XII.47})$$

Considering a scalar function φ as a composite function of the form

$$\varphi = \varphi [x^0 (x'^0, x'^1, x'^2, x'^3), x^1 (x'^0, x'^1, x'^2, x'^3), \dots]$$

we can write that

$$\frac{\partial \varphi}{\partial x'^\mu} = \sum_{\nu=0}^3 \frac{\partial \varphi}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu}$$

By (XII.41), $\partial \varphi / \partial x'^\mu$ is the μ -th covariant component of the gradient of φ calculated for the system K' [let us designate it by $(\nabla^* \varphi)_\mu'$], and $\partial \varphi / \partial x^\nu$ is the ν -th covariant component of $\nabla^* \varphi$ calculated for the system K [let us designate it by $(\nabla^* \varphi)_\nu$]. We can therefore write

$$(\nabla^* \varphi)_\mu' = \sum_{\nu=0}^3 \frac{\partial x^\nu}{\partial x'^\mu} (\nabla^* \varphi)_\nu \quad (\text{XII.48})$$

A comparison of transformations (XII.47) and (XII.48) shows that they are not identical. The role of the coefficients in one of them is played by the quantities $\partial x'^\mu / \partial x^\nu$, and in the other by the quantities $\partial x^\nu / \partial x'^\mu$.

Hence, vectors are said to be contravariant whose components transform according to the law (XII.47), i.e. like the components of

the vectors x^μ or dx^μ (like the differentials of the coordinates). Vectors are said to be covariant whose components transform according to the law (XII.48), i.e. like the components of a gradient (like the partial derivatives with respect to the coordinates).

We have seen above that if we are given a contravariant vector a^μ , we can always use the rule (XII.29) to introduce the covariant vector corresponding to it (and vice versa). Since the quantities a^μ and a_μ are close relatives, we shall treat them not as components of two different vectors, but as the contravariant and covariant components of the same vector.

For the Euclidean metric, the inverse transformation coefficients $\bar{\alpha}_\nu^\mu$ are related to the direct transformation coefficients α_ν^μ by the expression $\bar{\alpha}_\nu^\mu = \alpha_\mu^\nu$. Consequently, the direct and inverse transformations of coordinates have the form

$$x'^\mu = \sum_\nu \alpha_\nu^\mu x^\nu, \quad x^\nu = \sum_\mu \alpha_\nu^\mu x'^\mu$$

whence

$$\frac{\partial x'^\mu}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\mu}$$

Hence, when a space has a Euclidean metric, the coefficients of the transformations (XII.47) and (XII.48) coincide, and the difference between the contravariant and covariant vectors (and also the tensors) vanishes.

Transformation of Covariant Components. In a transition from the system K to the system K' , the covariant components of vectors obviously transform like their contravariant counterparts according to a linear law. Designating the coefficients of covariant component transformation by the symbol $\tilde{\alpha}_\mu^\nu$, we can write

$$a'_\mu = \sum_{\nu=0}^3 \tilde{\alpha}_\mu^\nu a_\nu \quad (\text{XII.49})$$

Let us find the relation between the coefficients of transformations of contravariant and covariant components, i.e. between the quantities α_ν^μ and $\tilde{\alpha}_\mu^\nu$. Raising or lowering of the index μ (or ν) on the component of a vector is attended by the multiplication of this component by $g_{\mu\mu}$ (or $g_{\nu\nu}$). Expression (XII.49) can therefore be transformed as follows:

$$a'_\mu g_{\mu\mu} = \sum_{\nu=0}^3 (\tilde{\alpha}_\mu^\nu g_{\nu\nu}) a_\nu \quad (\text{XII.50})$$

At the same time for contravariant components, we have

$$a'^\mu = \sum_{\nu=0}^3 \alpha_\nu^\mu a^\nu$$

Multiplying the left-hand and right-hand sides by $g_{\mu\mu}$, we obtain

$$a'^{\mu} g_{\mu\mu} = \sum_{\nu=0}^3 (\alpha_{\nu}^{\mu} g_{\mu\mu}) a^{\nu} \quad (\text{XII.51})$$

A comparison of formulas (XII.50) and (XII.51) shows that

$$\tilde{\alpha}_{\mu}^{\nu} g_{\nu\nu} = \alpha_{\nu}^{\mu} g_{\mu\mu} \quad (\text{XII.52})$$

With a view to (XII.4), it is simple to obtain that

$$\tilde{\alpha}_0^0 = \alpha_0^0, \quad \tilde{\alpha}_i^0 = -\alpha_i^0, \quad \tilde{\alpha}_0^i = -\alpha_i^0, \quad \tilde{\alpha}_i^k = \alpha_i^k \quad (i, k = 1, 2, 3) \quad (\text{XII.53})$$

A comparison of relations (XII.15) and (XII.52) allows us to write

$$\tilde{\alpha}_{\mu}^{\nu} = \bar{\alpha}_{\mu}^{\nu} \quad (\text{XII.54})$$

where $\bar{\alpha}_{\mu}^{\nu}$ are the coefficients of inverse transformation of the contravariant components.

We saw above that $\alpha_{\mu}^{\nu} = \partial x'^{\nu} / \partial x^{\mu}$ [see (XII.46)]. It is evident that

$$\bar{\alpha}_{\mu}^{\nu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}$$

Introducing this expression into (XII.54), we obtain coefficients for the direct transformation of the covariant components

$$\tilde{\alpha}_{\mu}^{\nu} = \bar{\alpha}_{\mu}^{\nu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}$$

This agrees with (XII.48).

Four-Dimensional Tensors. By a four-tensor of the second rank is meant a set of the 16 quantities $T^{\mu\nu}$ which in a transition from one coordinate system to another are transformed by the formula

$$T'^{\mu\nu} = \sum_{\rho, \sigma=0}^3 \alpha_{\rho}^{\mu} \alpha_{\sigma}^{\nu} T^{\rho\sigma} \quad (\text{XII.55})$$

where α_{ν}^{μ} are the coefficients from (XII.21) [compare with (X.10)]. In the inverse transformation

$$T^{\mu\nu} = \sum_{\rho, \sigma=0}^3 \bar{\alpha}_{\rho}^{\mu} \bar{\alpha}_{\sigma}^{\nu} T'^{\rho\sigma} \quad (\text{XII.56})$$

A particular case of a four-tensor is the tensor with the components

$$\Pi^{\mu\nu} = a^{\mu} b^{\nu} \quad (\text{XII.57})$$

where a^{μ} and b^{ν} are components of four-vectors.

The components of a four-tensor can be represented in three forms: as contravariant ones $T^{\mu\nu}$, covariant ones $T_{\mu\nu}$, and mixed

ones $T^\mu{}_\nu$. By analogy with (XII.57), raising or lowering of the time index does not change the sign of a component, while raising or lowering of a space index reverses the sign of a component [compare with (XII.29)]. Raising or lowering of an even number of space indices obviously leaves a component unchanged. Hence,

$$\begin{aligned} T_{00} &= T^{00}, \quad T_{01} = -T^{01}, \quad \dots, \quad T_{10} = -T^{10}, \quad T_{11} = T^{11}, \quad \dots \\ T^0{}_0 &= T^{00}, \quad T^0{}_1 = T^{01}, \quad \dots, \quad T^1{}_1 = -T^{01}, \quad T^1{}_0 = -T^{10}, \quad \dots \\ &\dots, \quad T^1{}_1 = T^1{}_1 = -T^{11}, \quad \dots \end{aligned} \quad (\text{XII.58})$$

Formulas for the transformation of the covariant components of the tensor (XII.57) can be obtained by taking (XII.49) into account:

$$\Pi'_{\mu\nu} = a'_\mu b'_\nu = \sum_\rho \tilde{\alpha}^\rho_\mu a_\rho \sum_\sigma \tilde{\alpha}^\sigma_\nu b_\sigma = \sum_{\rho, \sigma} \tilde{\alpha}^\rho_\mu \tilde{\alpha}^\sigma_\nu a_\rho b_\sigma = \sum_{\rho, \sigma} \tilde{\alpha}^\rho_\mu \tilde{\alpha}^\sigma_\nu \Pi_{\rho\sigma}$$

Similarly

$$T'_{\mu\nu} = \sum_{\rho, \sigma} \tilde{\alpha}^\rho_\mu \tilde{\alpha}^\sigma_\nu T_{\rho\sigma} \quad (\text{XII.59})$$

where the coefficients $\tilde{\alpha}^\nu_\mu$ are determined by the rules (XII.53).

In the same way, we can arrive at the formula

$$T^\mu{}_\nu = \sum_{\rho, \sigma} \alpha^\mu_\rho \tilde{\alpha}^\sigma_\nu T^\rho{}_\sigma \quad (\text{XII.60})$$

Examination of (XII.58) reveals that in general we must distinguish between the mixed components $T^\mu{}_\nu$ and $T_\nu{}^\mu$, i.e. see which index—the first or second—is a superscript and which is a subscript. Indeed, for instance, in the general case $T^1{}_0 = T^{10} \neq T^{01} = T_0{}^1$. For the symmetric tensors $S^{\mu\nu}$ (for which $S^{\mu\nu} = S^{\nu\mu}$), the mixed components $S^\mu{}_\nu$ and $S_\nu{}^\mu$ evidently coincide so that the indices may be placed one above the other. For antisymmetric tensors, the relation $A^\mu{}_\nu = -A_\nu{}^\mu$ holds.

In an antisymmetric four-tensor, six components are independent (four diagonal ones are zero, the others satisfy the condition $A^{\mu\nu} = -A^{\nu\mu}$). Consequently, the array of the components of an antisymmetric four-tensor is as follows:

$$(A^{\mu\nu}) = \begin{pmatrix} 0 & A^{01} & A^{02} & A^{03} \\ -A^{01} & 0 & A^{12} & A^{13} \\ -A^{02} & -A^{12} & 0 & A^{23} \\ -A^{03} & -A^{13} & -A^{23} & 0 \end{pmatrix} \quad (\text{XII.61})$$

Let us find formulas for transforming the components of the tensor (XII.61). By (XII.55) for the component A^{01} , we have

$$\begin{aligned} A'^{01} &= \sum_{\rho, \sigma=0}^3 \alpha_{\rho}^0 \alpha_{\sigma}^1 A^{\rho\sigma} = \sum_{\rho=0}^3 \alpha_{\rho}^0 \sum_{\sigma=0}^3 \alpha_{\sigma}^1 A^{\rho\sigma} \\ &= \alpha_0^0 \sum_{\sigma=0}^3 \alpha_{\sigma}^1 A^{0\sigma} + \alpha_1^0 \sum_{\sigma=0}^3 \alpha_{\sigma}^1 A^{1\sigma} + \alpha_2^0 \sum_{\sigma=0}^3 \alpha_{\sigma}^1 A^{2\sigma} + \alpha_3^0 \sum_{\sigma=0}^3 \alpha_{\sigma}^1 A^{3\sigma} \end{aligned}$$

The coefficients α_2^0 and α_3^0 are zeros [see (XII.21)], therefore the last two sums will vanish. Of the four coefficients α_{σ}^1 , only two are non-zero: α_0^1 and α_1^1 . Consequently, in the first two sums, only the first two addends are non-zero. Hence,

$$A'^{01} = \alpha_0^0 (\alpha_0^1 A^{00} + \alpha_1^1 A^{01}) + \alpha_1^0 (\alpha_0^1 A^{10} + \alpha_1^1 A^{11})$$

Taking into account that $A^{00} = A^{11} = 0$, $A^{10} = -A^{01}$, substituting α_0 for α_0^0 and α_1^1 , and also α_1 for α_1^0 and α_0^1 , we arrive at the formula

$$A'^{01} = (\alpha_0^2 - \alpha_1^2) A^{01} = A^{01}$$

[see relation (XII.22)]. In a similar way, we can obtain transformation formulas for the other components of the tensor $A^{\mu\nu}$.

The formulas for the transformation of all six independent components are as follows:

$$\begin{aligned} A'^{01} &= A^{01}, \quad A'^{02} = \alpha_0 A^{02} + \alpha_1 A^{12}, \quad A'^{03} = \alpha_0 A^{03} + \alpha_1 A^{13} \\ A'^{12} &= \alpha_0 A^{12} + \alpha_1 A^{02}, \quad A'^{13} = \alpha_0 A^{13} + \alpha_1 A^{03}, \quad A'^{23} = A^{23} \end{aligned} \quad (\text{XII.62})$$

We must note that the formulas for the inverse transformation differ from formulas (XII.62) only in the sign of the terms containing the factor α_1 [see (XII.23)].

We need formulas (XII.62) in electrodynamics.

Let us form the following expression from the components of the tensor $\Pi^{\mu\nu} = a^{\mu}b^{\nu}$:

$$\sum_{\mu} \Pi^{\mu}_{\mu} = \sum_{\mu} a^{\mu}b_{\mu}$$

This expression is an invariant, i.e. a scalar. Similarly, for any tensor $T^{\mu\nu}$, the expression

$$\sum_{\mu} T^{\mu}_{\mu} = T^0_0 + T^1_1 + T^2_2 + T^3_3, \quad (\text{XII.63})$$

is a scalar [compare with (X.21)]. It is called the trace of the tensor. A glance at (XII.33) shows that $\sum \Pi^{\mu}_{\mu} = \sum \Pi_{\mu}^{\mu}$. Similarly

$$\sum_{\mu} T^{\mu}_{\mu} = \sum_{\mu} T_{\mu}^{\mu} \quad (\text{XII.64})$$

The scalar product of the four-vector a^ν and the four-tensor $T^{\mu\nu}$ is defined to be the four-vector b^μ whose components are determined by the formula

$$b^\mu = \sum_{\nu} T^{\mu\nu} a^\nu \quad (\text{XII.65})$$

Invariant Four-Tensors. A tensor which when multiplied by a vector leaves the latter unchanged is naturally called a **unit tensor**. The components of this tensor must obviously be taken equal to

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \quad (\text{XII.66})$$

Indeed, the introduction of these values into formula (XII.65) yields

$$b^\mu = \sum_{\nu} \delta^\mu_\nu a^\nu = a^\mu$$

as matters should be for a unit tensor.

It follows from (XII.66) that $\delta^\mu_\nu = \delta^\nu_\mu$, i.e. that the tensor δ^μ_ν is symmetric. This is why we have arranged the indices one over the other. The trace of the tensor δ^μ_μ is

$$\sum_{\mu=0}^3 \delta^\mu_\mu = 4$$

The contravariant and covariant components of a unit tensor are customarily designated by the symbols $g^{\mu\nu}$ and $g_{\mu\nu}$. It can be seen that

$$(g^{\mu\nu}) = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{XII.67})$$

A comparison with (XII.4) shows that the tensor $g^{\mu\nu}$ (like $g_{\mu\nu}$) is a metric tensor.

Let us form the expression

$$\sum_{\nu} g_{\mu\nu} a^\nu$$

Taking into account (XII.67), we find that when $\mu = 0$, this expression is $a^0 = a_0$, while when $\mu = i$ ($i = 1, 2, 3$), we obtain $-a^i = a_i$. It thus follows that

$$\sum_{\nu} g_{\mu\nu} a^\nu = a_\mu$$

Similarly, we can see that

$$\sum_{\nu} g^{\mu\nu} a_{\nu} = a^{\mu}$$

Consequently, the scalar product of the vectors a^{μ} and b^{ν} can be written in any of the following three ways:

$$\sum_{\mu} a^{\mu} b_{\mu} = \sum_{\mu, \nu} g_{\mu\nu} a^{\mu} b^{\nu} = \sum_{\mu, \nu} g^{\mu\nu} a_{\mu} b_{\nu} \quad (\text{XII.68})$$

The tensors δ_{ν}^{μ} , $g_{\mu\nu}$, and $g^{\mu\nu}$ are invariant—their components are identical in all coordinate systems. The absolutely antisymmetric unit four-pseudotensor of the fourth rank, $\varepsilon^{\mu\nu\rho\sigma}$, is also invariant. The components of this tensor are determined similarly to those of the pseudotensor ε_{ijkl} in Euclidean space [see (VI.15)]. If at least two indices coincide, $\varepsilon^{\mu\nu\rho\sigma}$ is zero. Therefore, of its $4^4 = 256$ components, only $4! = 24$ components are non-zero. It is assumed that

$$\varepsilon^{0123} = +1 \quad (\text{XII.69})$$

The remaining 23 components are assumed to equal $+1$ or -1 depending on what number of permutations of two indices—even or odd—gives us the sequence μ, ν, ρ, σ from the sequence 0, 1, 2, 3. It is obvious that 12 components have the value $+1$ and 12 the value -1 .

According to the rule we have adopted above, lowering of all the non-zero indices in $\varepsilon^{\mu\nu\rho\sigma}$ must change the sign of the relevant component. Therefore, $\varepsilon_{0123} = -1$. Similarly, $\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma}$ (of the four indices, one must be 0, and the other three—1, 2, 3). It thus follows that

$$\sum_{\mu, \nu, \rho, \sigma} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} = -24 \quad (\text{XII.70})$$

The Ostrogradsky-Gauss Theorem. In Euclidean space, the Ostrogradsky-Gauss theorem is written as follows [see (XII.13)]:

$$\int_V \sum_{i=1}^3 \frac{\partial a_i}{\partial x_i} dV = \oint_f \sum_{i=1}^3 a_i df_i \quad (\text{XII.71})$$

(the integral of the divergence of the vector \mathbf{a} over a certain volume V equals the flux of this vector through the surface f enclosing V). The quantities df_i are components of the vector $d\mathbf{f} = \mathbf{n} df$. They have the values $df_x = dy dz$, $df_y = dz dx$, $df_z = dx dy$.

The following relation is a generalization of the Ostrogradsky-Gauss theorem for pseudo-Euclidean four-space:

$$\int \sum_{\mu=0}^3 \frac{\partial a^{\mu}}{\partial x^{\mu}} dV^* = \oint \sum_{\mu=0}^3 a^{\mu} df_{\mu} \quad (\text{XII.72})$$

where a^μ is a four-vector, $dV^* = dx^0 dx^1 dx^2 dx^3 = c dt dV$ is an element of the volume in four-space, df^μ is a component of the four-vector of an element of the hypersurface enclosing the four-volume over which the integral on the left-hand side of formula (XII.72) is taken. The components df^μ have the values $df^0 = dx^1 dx^2 dx^3 = dV$, $df^1 = dx^0 dx^2 dx^3 = c dt dy dz$, etc.

Let us use the Ostrogradsky-Gauss theorem to prove a relation that we need in Sec. 40. To make our proof more obvious, we shall

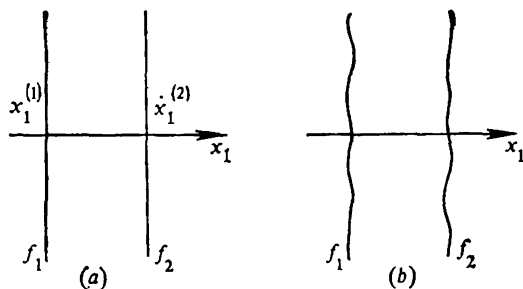


Fig. XII.1.

first give it for Euclidean three-space, and then perform similar calculations for four-space.

Assume that we have the vector a_i whose divergence is zero:

$$\sum_{i=1}^3 \frac{\partial a_i}{\partial x_i} = 0$$

Let the vector a_i be non-zero in a restricted region of space. By (XII.71), we have

$$\int \sum_i \frac{\partial a_i}{\partial x_i} dV = \oint_f \sum_i a_i df_i = 0 \quad (\text{XII.73})$$

Equation (XII.73) holds for an arbitrarily taken volume V and the surface f enclosing it. Let us take as V the volume confined between two infinite planes $x_1 = x_1^{(1)} = \text{const}$ and $x_1 = x_1^{(2)} = \text{const}$ (Fig. XII.1a). The integral over the side surface of this volume is zero because, by assumption, at infinity $a_i = 0$. Consequently, the right-hand side of formula (XII.73) can be written as (for the plane

$$\int_{(1)} a_1 df_1 - \int_{(2)} a_1 df_1 = 0$$

$x_1 = \text{const}$, the components df_2 and df_3 are zero). Hence

$$\int_{(1)} a_i df_i = \int_{(2)} a_i df_i = \text{const}$$

The result we have obtained signifies that the integral $\int a_i df_i$ taken over any infinite plane $x_1 = \text{const}$ has an identical value. We must note that the coordinates x_2 and x_3 take on all values from $-\infty$ to $+\infty$ in integration.

Now let us take as V a volume enclosed by two surfaces of an arbitrary shape for all of whose points the coordinate x_1 is finite, while the coordinates x_2 and x_3 vary from $-\infty$ to $+\infty$. Consequently, the edges of the surfaces are at infinity (Fig. XII.1b). In this case, the right-hand side of formula (XII.73) can be written as

$$\int_{f_1} \sum_i a_i df_i - \int_{f_2} \sum_i a_i df_i = 0$$

whence

$$\int \sum_i a_i df_i = \text{const} \quad (\text{XII.74})$$

i.e. has the same value for any surface including all the two-dimensional space $x_2 x_3$ (the space yz).

Now let us assume that there is a tensor T_{ik} satisfying the condition

$$\sum_{k=1}^3 \frac{\partial T_{ik}}{\partial x_k} = 0 \quad (i = 1, 2, 3) \quad (\text{XII.75})$$

The components of T_{ik} differ from zero in a restricted region of space

Let us form the vector a having the components

$$a_k = \sum_i T_{ik} b_i \quad (\text{XII.76})$$

where b_i are the components of an arbitrary constant vector ($b_i = \text{const}$). The divergence of the vector a will be zero. Indeed,

$$\sum_k \frac{\partial a_k}{\partial x_k} = \sum_k \frac{\partial}{\partial x_k} \sum_i T_{ik} b_i = \sum_i b_i \sum_k \frac{\partial T_{ik}}{\partial x_k} = 0$$

Therefore, the vector (XII.76) satisfies the conditions in which relation (XII.74) is observed. Substitution of the values (XII.76) into (XII.74) yields

$$\int \sum_k a_k df_k = \int \sum_k \left(\sum_i T_{ik} b_i \right) df_k = \sum_i b_i \int \sum_k T_{ik} df_k = \text{const} \quad (\text{XII.77})$$

(integration is performed over the surface including the entire two-dimensional space x_2x_3).

Introducing the symbol

$$p_i = \int \sum_k T_{ik} df_k \quad (\text{XII.78})$$

we can write relation (XII.77) as

$$\sum_i b_i p_i = \text{const}$$

Because of the constancy and arbitrariness of the quantities b_i , the last expression can be observed only provided that all the p_i 's are also constant.

We have thus arrived at the following statement: if the tensor T_{ik} satisfies condition (XII.75) and its components are non-zero in a restricted region of space, the values of the vector components p_i do not depend on over which of the surfaces confining the entire two-dimensional space x_2x_3 integration in formula (XII.78) is performed.

Now let us reason similarly for four-space. Assume that there is the vector a^ν whose four-divergence is zero:

$$\sum_{\nu=0}^3 \frac{\partial a^\nu}{\partial x^\nu} = 0 \quad (\text{XII.79})$$

(a^ν are non-zero in a restricted region of four-space). By (XII.72), we have

$$\int \sum_\nu \frac{\partial a^\nu}{\partial x^\nu} dV^* = \oint \sum_\nu a^\nu df_\nu = 0 \quad (\text{XII.80})$$

This equation holds for any closed hypersurface; the integral on the left-hand side is evaluated over the four-volume confined by this hypersurface. Let us take as this volume the part of four-space confined between two infinite hyperplanes $x^0 = x^0_{(1)} = \text{const}$ and $x^0 = x^0_{(2)} = \text{const}$. The coordinates x^1, x^2, x^3 for points of such hyperplanes vary from $-\infty$ to $+\infty$. Consequently, each of the hyperplanes consists of the whole three-dimensional space taken at the instant $t_1 = x^0_{(1)}/c$ for the first plane and at the instant $t_2 = x^0_{(2)}/c$ for the second one.

For the chosen four-volume, the right-hand side of relation (XII.80) can be written as

$$\int_{(1)} a^0 df_0 - \int_{(2)} a^0 df_0 = 0$$

(since $dx^0 = 0$, all the df^i 's are zero). Hence

$$\int a^0 df_0 = \text{const} \quad (\text{XII.81})$$

This means that the value of the integral (XII.81) does not depend on which of the hyperplanes $x^0 = \text{const}$ integration is performed over.

Now let us take a four-volume enclosed by two hypersurfaces of an arbitrary shape for all of whose points the coordinate x^0 is finite, and the coordinates x^1, x^2, x^3 take on values from $-\infty$ to $+\infty$. Such hypersurfaces include the entire three-dimensional space. Writing the right-hand side of relation (XII.80) for such a volume, we have

$$\int_{(1)} \sum_v a^v df_v - \int_{(2)} \sum_v a^v df_v = 0$$

Hence, the integral

$$\int \sum_v a^v df_v = \text{const} \quad (\text{XII.82})$$

i.e. does not depend on which of the hypersurfaces enclosing the entire three-dimensional space it is taken over. In other words, this integral is time-independent, and its value is conserved.

Let us take as a^v the four-vector with the components

$$a^v = \sum_{\mu} T^{\mu v} b_{\mu} \quad (\text{XII.83})$$

where b_{μ} are the components of an arbitrary constant four-vector ($b_{\mu} = \text{const}$), and $T^{\mu v}$ is a four-tensor satisfying the condition

$$\sum_{v=0}^3 \frac{\partial T^{\mu v}}{\partial x^v} = 0 \quad (\mu = 0, 1, 2, 3) \quad (\text{XII.84})$$

The components of this tensor are assumed to be non-zero in a restricted region of four-space. It is not difficult to see that the vector (XII.83) satisfies condition (XII.79). Consequently, relation (XII.82) must be observed for it. Substitution of the values (XII.83) into (XII.82) yields

$$\int \sum_v a^v df_v = \int \sum_v \left(\sum_{\mu} T^{\mu v} b_{\mu} \right) df_v = \sum_{\mu} b_{\mu} \int \sum_v T^{\mu v} df_v = \text{const}$$

(integration is performed over an arbitrary hypersurface including the entire three-dimensional space).

If we introduce the symbol

$$p^{\mu} = \int \sum_v T^{\mu v} df_v \quad (\text{XII.85})$$

the relation we have obtained can be written as

$$\sum_{\mu} b_{\mu} p^{\mu} = \text{const}$$

Owing to the arbitrariness and constancy of the quantities b_μ , the latter condition can be observed only provided that all the p^μ 's are constant (i.e. time-independent). We thus arrive at the following conclusion. *If there is a tensor $T^{\mu\nu}$ whose components are non-zero only in a restricted region of four-space and satisfy condition (XII.84), the components of the four-vector (XII.85) are conserved, i.e. do not change their value with time.* It is obvious that the four-vector with the components

$$p^\mu = \alpha \int \sum_{\nu} T^{\mu\nu} df_\nu \quad (\text{XII.86})$$

where α is an arbitrary constant, will also be conserved.

XIII. The Dirac Delta Function

The Dirac delta function (δ function) is defined as the function determined as follows: $\delta(x) = 0$ at all x 's differing from zero; at $x = 0$, the function $\delta(x)$ becomes infinite, and so that

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1 \quad (\text{XIII.1})$$

The delta function is useful owing to its following property:

$$\int_{-\infty}^{+\infty} f(x) \delta(x) dx = f(0) \quad (\text{XIII.2})$$

where $f(x)$ is an arbitrary continuous function of x . This property follows from the definition of the delta function. Indeed, since $\delta(x) = 0$ at all $x \neq 0$, only the vicinity of the point $x = 0$ makes a non-zero contribution to the integral (XIII.2). In this vicinity, $f(x)$ can be assumed equal to $f(0)$. Putting $f(0)$ outside the integral sign and taking (XIII.1) into account, we arrive at (XIII.2).

It is evident that the function $\delta(x - a)$ has the same properties in the vicinity of the point $x = a$ as the function $\delta(x)$ in the vicinity of the point $x = 0$. Particularly,

$$\int_{-\infty}^{+\infty} f(x) \delta(x - a) dx = f(a) \quad (\text{XIII.3})$$

The integration region in formulas (XIII.2) and (XIII.3) must not necessarily extend from $-\infty$ to $+\infty$, it is sufficient for this region to include a singular point at which the delta function is non-zero.

Like our introduction of $\delta(x)$, a three-dimensional delta function designated by $\delta(\mathbf{r})$ is defined. It is zero everywhere except for the

origin of coordinates. At the origin of coordinates, $\delta(\mathbf{r})$ becomes infinite so that

$$\int \delta(\mathbf{r}) dV = 1 \quad (\text{XIII.4})$$

The integral is evaluated over the entire three-dimensional space.

The three-dimensional delta function can be written as the product of three one-dimensional delta functions:

$$\delta(\mathbf{r}) = \delta(x) \delta(y) \delta(z) \quad (\text{XIII.5})$$

It follows from the definition of the delta function that

$$\int f(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_0) dV = f(\mathbf{r}_0) \quad (\text{XIII.6})$$

The integration region in formula (XIII.6) must not necessarily include the entire three-dimensional space, it is sufficient that this region include the point determined by the vector \mathbf{r}_0 .

XIV. The Fourier Series and Integral

A function satisfying the condition

$$f(t + T) = f(t) \quad (\text{XIV.1})$$

where T is a constant, is said to be periodic¹. The quantity T is called the **period** of the function. A very simple example of a periodic function is the harmonic function $f(t) = a \cos(\omega t + \alpha)$, where $\omega = 2\pi/T$ is the cyclic (angular) frequency of the function.

The overwhelming majority of periodic functions encountered in physical problems can be written as the series

$$f(t) = \sum_{n=0}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad (\text{XIV.2})$$

For brevity, we have used the notation

$$\omega_0 = \frac{2\pi}{T} \quad (\text{XIV.3})$$

where T is the period of the function.

The series (XIV.2) is called the **Fourier series**. The constants a_n and b_n are called Fourier coefficients. We shall not discuss the conditions which a function must satisfy for its values to coincide with those of the series (XIV.2), referring readers interested in this matter to texts on calculus.

A non-periodic function can also be represented as a Fourier series. But such a representation will be suitable for the non-periodic function only on the segment from $-T/2$ to $+T/2$.

¹Having in mind the applications, we have designated the independent variable by t instead of by x .

Expression (XIV.2) is an expansion of the function $f(t)$ into a series in the functions

$$1, \cos \omega_0 t, \sin \omega_0 t, \dots, \cos n\omega_0 t, \sin n\omega_0 t, \dots \quad (\text{XIV.4})$$

The system of functions (XIV.4) is orthogonal over the segment $-T/2, +T/2$. This signifies that an integral of the product of two different functions of the system over this segment is zero, while a similar integral of the square of any function is non-zero. The property of orthogonality can be written briefly as

$$\int_{-T/2}^{+T/2} \psi_n \psi_m dt = \delta_{nm} q_n$$

Here ψ_n and ψ_m are any functions belonging to the system (XIV.4). It is a simple matter to see that $q_0 = T$ and $q_n = T/2$ (for $n \neq 0$). For this end, it is sufficient to recall that the average values of the square of a cosine and the square of a sine are $1/2$.

The orthogonality of the system of functions (XIV.4) allows us to find the values of the Fourier coefficients. Let us multiply relation (XIV.2) by the function $\cos m\omega_0 t$ (here $m \neq 0$) and then integrate over the segment $-T/2, +T/2$. Owing to orthogonality, of all the integrals on the right-hand side, only one will be non-zero, namely,

$$\int_{-T/2}^{+T/2} a_m \cos^2 m\omega_0 t dt = a_m \frac{T}{2}$$

We therefore arrive at the formula $\int_{-T/2}^{+T/2} f(t) \cos m\omega_0 t dt = a_m T/2$,

whence

$$a_m = \frac{2}{T} \int_{-T/2}^{+T/2} f(t) \cos m\omega_0 t dt \quad (m \neq 0) \quad (\text{XIV.5})$$

Similar calculations lead to expressions for b_m and a_0 :

$$b_m = \frac{2}{T} \int_{-T/2}^{+T/2} f(t) \sin m\omega_0 t dt \quad (m \neq 0) \quad (\text{XIV.6})$$

$$a_0 = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) dt, \quad b_0 = 0 \quad (\text{XIV.7})$$

The Fourier series can be written in the complex form. For this purpose, we replace the cosine and sine with exponentials:

$$\cos n\omega_0 t = \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2}, \quad \sin n\omega_0 t = \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}$$

The expression in parentheses in formula (XIV.2) now becomes

$$a_n \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2} + b_n \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}$$

$$= \frac{a_n + ib_n}{2} e^{-in\omega_0 t} + \frac{a_n - ib_n}{2} e^{in\omega_0 t} = C_n e^{-in\omega_0 t} + C_{-n} e^{in\omega_0 t} \quad (\text{XIV.8})$$

We have introduced the following coefficients instead of a_0 , a_n , and b_n :

$$C_0 = a_0, \quad C_n = \frac{a_n + ib_n}{2}, \quad C_{-n} = \frac{a_n - ib_n}{2} \quad (n \neq 0) \quad (\text{XIV.9})$$

We must note that

$$C_{-n} = C_n^* \quad (\text{XIV.10})$$

(the asterisk stands for a complex conjugate quantity).

Making the substitution (XIV.8) in (XIV.2), we obtain

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{-in\omega_0 t} \quad (\text{XIV.11})$$

Expression (XIV.11) is an expansion of the function $f(t)$ into a series in the functions

$$1, e^{\pm i\omega_0 t}, e^{\pm i2\omega_0 t}, \dots \quad (\text{XIV.12})$$

The system of functions (XIV.12) is also orthogonal over the segment $-T/2, +T/2$. For complex functions, this signifies that

$\int_{-T/2}^{+T/2} \psi_n \psi_m^* dt = \delta_{nm} q_n$ (here $q_n \neq 0$). Indeed, it is not difficult to see that

$$\int_{-T/2}^{+T/2} e^{i(n-m)\omega_0 t} dt = \delta_{nm} T \quad (\text{XIV.13})$$

The orthogonality of the system of functions (XIV.12) allows us to find the values of the coefficients C_n . Let us multiply relation (XIV.11) by $e^{im\omega_0 t}$ and then integrate over the segment $-T/2, +T/2$. Owing to orthogonality, of all the integrals on the right-hand side, only one will be non-zero, namely

$$\int_{-T/2}^{+T/2} C_m e^{-i(n-m)\omega_0 t} dt = C_m T$$

We therefore arrive at the formula $\int_{-T/2}^{+T/2} f(t) e^{im\omega_0 t} dt = C_m T$,

If two functions φ and f are related by an expression of the form

$$\varphi(x) = \int_a^b f(t) K(x, t) dt \quad (\text{XIV.23})$$

the function $\varphi(x)$ is said to be the integral transform of the function $f(t)$, and the function $K(x, t)$, the kernel of the integral transform.

A comparison of formulas (XIV.21) and (XIV.23) shows that the function $\varphi(\omega)$ determined by expression (XIV.21) is the integral transform of the function $f(\xi)$, the kernel of the transform being $K(\omega, \xi) = (1/\sqrt{2\pi}) e^{i\omega\xi}$. This is why the function $\varphi(\omega)$ is called the **Fourier transform** of the function $f(\xi)$. The function $f(t)$ determined by expression (XIV.22) is called the **inverse Fourier transform**. We must note that the direct [formula (XIV.21)] and the inverse [formula (XIV.22)] Fourier transforms differ only in the sign of the exponent.

The function $f(\xi)$ in expression (XIV.21) is also called the Fourier **image** of the function $\varphi(\omega)$.

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Igor V. Savelyev, D.Sc. (Physics and Mathematics), has been head of the Department of General Physics at the Moscow Institute of Engineering Physics for about 25 years, after devoting quite a few years to experimental physics. He is the author of about 100 published works in physics, including **Physics. A General Course**, in three volumes, translated into English by Mir Publishers, and **A Collection of Questions and Problems in General Physics** (in Russian). His principal interests include instruction in physics at higher technical schools and its improvement. Professor Savelyev holds the title of Honoured Scientist of the RSFSR and is a USSR State Prize Winner.